George Long

Oxford Review Series

PLANE GEOMETRY

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PLANE GEOMETRY

BY

GEORGE F. WILDER, M.A.

Head of the Department of Mathematics, John Adams High School, Long Island, N. Y. Author, DRILL EXERCISES IN PLANE GEOMETRY

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PREFACE

THE aim of this book is to present a comprehensive and pedagogically sound review of plane geometry in such a manner as to enable the pupil to obtain a view of the subject as a whole. Hence the arrangement of propositions by books and in strictly logical order is left to the textbook, and the arrangement here adopted is a topical grouping in accordance with the *tool* value of the proposition.

The following features recommend this book to teachers of mathematics:

I. It is up to date.—The content meets the very latest requirements for the subject as prescribed by the Board of Regents for the State of New York and by the College Entrance Examination Board for the country at large.

II. The grouping of the propositions is such as to make for unity in the pupil's knowledge of the subject and to strengthen his ability in solving "originals."

III. In each proposition, the reasons for the various steps in the proof are left to the pupil, thus exacting thought and discouraging memorizing.

IV. Exercises, carefully culled from recent examination papers, have been placed at the end of each group of propositions to enable the pupil to test and apply his knowledge of the preceding material.

V. Complete recent examination papers, set by the University of the State of New York, show the trend of present examinations in the subject, provide additional drill material, and familiarize the pupil with the type of question he will have to face in the final examination.

The author is greatly indebted to Mr. Elmer Schuyler, Head of the Department of Mathematics, Bay Ridge High School, Brooklyn, N. Y., for his excellent criticism and suggestions and for his careful reading of the manuscript and proofs.

G. F. W.

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PLANE GEOMETRY

I. INTRODUCTION

DEFINITIONS

Acute angle.—An angle less than a right angle.

Acute triangle.—A triangle in which each angle is acute.

Adjacent angles.—Two angles having a common vertex and a common side between them.

Alternation.—Forming a new proportion from a given proportion by taking the terms in the following order: *first, third, second, fourth.*

Altitude of a parallelogram.—The perpendicular distance between two parallel sides.

Altitude of a trapezoid.—The perpendicular distance between the parallel sides.

Altitude of a triangle.—The perpendicular from a vertex to the opposite side (produced if necessary).

Angle inscribed in a segment.—An angle whose vertex is on the arc of the segment, and whose sides pass through the ends of this arc.

Antecedents.—The first terms of two or more equal ratios.

Apothem of a regular polygon.—The radius of the inscribed circle.

Arc.—Any part of a circle.

Axiom.—A statement whose truth is assumed without proof.

Base of a triangle (or **parallelogram.**)—The side on which the figure is assumed to rest. Any side may be considered as the base.

Bases of a trapezoid.—The parallel sides.

Center of a regular polygon.—The center of the circumscribed circle.

Central angle.—An angle formed by two radii.

Chord.—A line joining two points on a circle.

Circle.—A plane closed curve all points of which are equally distant from a fixed point within called the center.

Circumference.—The length of a circle.
Circumscribed circle.—A circle passing through each vertex of a polygon.
Circumscribed polygon.—A polygon each side of which is tangent to a circle.
Complementary angles.—Two angles whose sum is a right angle.
Concentric circles.—Circles having the same center.
Conclusion.—That part of a proposition which is to be proved.
Concurrent lines.—Three or more lines passing through one point.
Congruent figures.—Figures that can be made to coincide.
Consequents.—The second terms of two or more equal ratios.
Constant.—A quantity whose value remains fixed.
Converse of a proposition.—A proposition obtained by interchanging the hypothesis and the conclusion of a given proposition.
Corollary.—A theorem easily derived from another theorem.
Corresponding parts.—See **homologous parts.**

Decagon.—A polygon having ten sides.
Degree (angle).—One ninetieth ($\frac{1}{90}$) of a right angle.
Degree (arc).—If a circle is divided into 360 equal parts, each part is an arc of one degree.
Demonstration.—See **Proof.**
Diagonal of a polygon.—Any line, except a side, joining two vertices.
Diameter.—A chord passing through the center of a circle.
Distance from a point to another point.—The length of the line joining the two points.
Distance from a point to a line.—The length of the perpendicular from the point to the line.
Dodecagon.—A polygon having twelve sides.

Equal figures.—See **Equivalent figures.**
Equiangular.—Having equal angles.
Equilateral.—Having equal sides.
Equivalent figures.—Figures having equal areas.
Exterior angle of a polygon.—An angle formed by a side and the prolongation of an adjacent side.
External common tangent.—That common tangent to two circles which crosses the prolongation of the line joining the two centers.
Extremes of a proportion.—The first and last terms.

Foot of a perpendicular.—The point common to a perpendicular and the line to which it is drawn.
Fourth proportional.—The fourth term of a proportion.

Geometry (plane).—Treats of figures formed by points and lines lying in the same plane.

Hexagon.—A polygon having six sides.
Homologous parts of congruent figures.—Those lines and angles which would coincide if the figures were superposed.
Homologous parts of similar figures.—The equal angles; also, sides similarly situated with respect to the equal angles.
Hypotenuse.—The side of a right triangle opposite the right angle.
Hypothesis.—The conditional part of a proposition; that which is assumed to be true.

Inscribed angle.—An angle whose sides are chords drawn from the same point on a circle.
Inscribed circle.—The circle to which each side of a polygon is tangent.
Inscribed polygon.—A polygon all of whose vertices are on a circle.
Internal common tangent.—That common tangent to two circles which crosses the line joining the two centers.
Internal division of a line.—Any point in a line is said to be a point of internal division.
Isosceles trapezoid.—A trapezoid in which the non-parallel sides are equal.
Isosceles triangle.—A triangle which has two equal sides.

Limit of a variable.—A constant such that the difference between it and the variable can be made to become and remain less than any given value.
Line of centers.—The line determined by the centers of two circles.
Locus of points.—One or more lines such that all points on the lines satisfy a given condition, and all points that satisfy a given condition are on the line or lines.

Mean proportional.—If the second and third terms of a proportion are equal, either is said to be the mean proportional between the first and fourth. Thus, in the proportion $a:b=b:c$, b is a mean proportional between a and c.

Means.—The second and third terms of a proportion.
Median of a trapezoid.—The line joining the midpoints of the non-parallel sides.
Median of a triangle.—A line from any vertex to the midpoint of the opposite side.

Oblique angle.—Any angle that is neither a right angle nor a straight angle.
Oblique lines.—Lines that are not perpendicular to each other.
Obtuse angle.—An angle greater than a right angle and less than a straight angle.
Obtuse triangle.—A triangle having an obtuse angle.
Octagon.—A polygon having eight sides.
Opposite angles.—See **Vertical angles.**
Opposite proposition.—A proposition whose hypothesis and conclusion are respectively the negatives of the hypothesis and conclusion of a given proposition.

Parallel lines.—Lines lying in the same plane that do not meet however far prolonged.
Parallelogram.—A quadrilateral whose opposite sides are parallel.
Pentagon.—A polygon having five sides.
Perimeter.—The sum of the sides of a polygon.
Perpendicular lines.—Lines that meet at right angles.
Pi (π).—The constant ratio of the circumference of a circle to its diameter.
Plane.—A surface such that a line joining any two points in it lies entirely in the surface.
Point of contact.—The point common to a circle and a tangent.
Polygon.—A closed broken line.
Postulate.—A construction whose possibility is assumed.
Problem of computation.—From given numerical values of certain parts of a figure, to calculate the numerical values of other parts.
Problem of construction.—To draw with ruler and compasses a figure satisfying given conditions.
Projection of a line on another line.—That part of the second line which is included between the perpendiculars from the ends of the first line.
Projection of a point on a line.—The foot of the perpendicular from the point to the line.
Proof.—A course of reasoning which shows the truth of a proposition.

Proportion.—A statement of equality between two or more ratios.

Quadrant.—One fourth of a circle.
Quadrilateral.—A polygon having four sides.

Radius.—A line drawn from the center of a circle to any point on the circle.

Radius of a regular polygon.—The radius of the circumscribed circle.
Ratio.—An indicated quotient of one number by another.
Rectangle.—A parallelogram whose angles are right angles.
Regular polygon.—A polygon which is both equiangular and equilateral.
Rhombus.—An equilateral parallelogram.
Right angle.—An angle equal to one-half a straight angle.
Right triangle.—A triangle containing a right angle.

Scalene triangle.—A triangle having no equal sides.
Secant.—A line cutting a circle in two points.
Sector.—The figure formed by an arc and the radii drawn to its extremities.
Segment of a circle.—The figure formed by an arc and its chord.
Segment of a line.—The part of the line included between two definite points on the line.
Similar arcs.—Arcs which have equal central angles.
Similar polygons.—Polygons in which the corresponding angles are equal and the corresponding sides are proportional.
Similar sectors.—Sectors having equal central angles.
Similar segments.—Segments having equal central angles.
Square.—An equilateral rectangle.
Straight angle.—An angle whose sides extend in opposite directions and form a straight line.
Supplementary-adjacent angles.—Two angles which are both supplementary and adjacent.
Supplementary angles.—Two angles whose sum is a straight angle.

Tangent.—A line that has one point in common with a circle, and all other points on the line are outside of the circle.
Tangent circles.—Circles tangent to the same line at the same point.
Theorem.—A statement to be proved.

Transform.—To change the shape of a figure without changing its area.

Transversal.—A line intersecting two or more other lines.

Trapezoid.—A quadrilateral having one and only one pair of sides parallel.

Variable.—A quantity which may have different values during the same discussion.

Vertex angle of a triangle.—The angle opposite the base of the triangle.

Vertical angles.—Two angles such that the sides of one are the prolongations of the sides of the other.

AXIOMS AND POSTULATES

1. Things equal to the same thing, or to equal things, are equal to each other.
2. If equals are added to equals, the sums are equal.
3. If equals are subtracted from equals, the remainders are equal.
4. If equals are multiplied by equals, the products are equal.
   (a) Doubles of equals are equal.
5. If equals are divided by equals, the quotients are equal.
   (a) Halves of equals are equal.
6. If unequals are subtracted from equals, the remainders are unequal in the opposite order.
7. If the first of three quantities is greater than the second, and the second is greater than the third, then the first is greater than the third.
8. The whole is greater than any of its parts.
9. The whole is equal to the sum of all its parts.
10. Any quantity may be replaced by an equal quantity in any equation or in an inequality. (*Substitution*)
11. Any geometric figure may be moved from one position to another without change of size or form. (*Superposition axiom*)
12. A straight line may be produced to any required length.
13. Two points determine a line; or, through two points one and only one straight line can be drawn.
14. A straight line is the shortest line between two points.
15. Two straight lines can intersect in only one point.
16. Two intersecting lines cannot both be parallel to a third straight line.
17. A circle may be drawn having any point as center and any line as radius.

SUGGESTIONS TO THE STUDENT

You will find the following suggestions helpful in solving original exercises:

1. Draw an accurate figure.
2. Do not draw a figure for a special case.
   (a) Do not draw an isosceles triangle or an equilateral triangle unless the hypothesis requires such a triangle.
   (b) Do not draw lines perpendicular to each other, nor through special points such as midpoints, centers, etc., unless the hypothesis states that they are to be perpendicular or to pass through such points.
   (c) Remember that the words "any point" or "any line" in the statement of a proposition mean *every* such point or line.
3. Write accurately the hypothesis and the conclusion in terms of the letters on the figure.
4. Recall as many propositions as you can that apply to the figure, and select those that seem to have a bearing on your problem.
5. Analyze your proposition.
   (a) Assume the proposition to be true, and, from this assumption, try to obtain some fact already known to be true.
   (b) Try to obtain a proof by considering the steps of the analysis in the reverse order, proceeding from the known fact to the unknown.
6. As soon as facts concerning a figure are known, indicate them in some way on the figure itself.

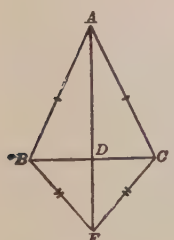


FIG. A

FIG. B

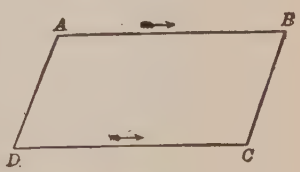
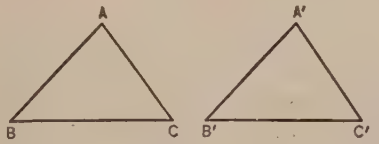


FIG. C

(a) Equal lines may be indicated as in Fig. A.
(b) Equal angles may be indicated by arcs as in Fig. B.
(c) Parallel lines may be indicated as in Fig. C.

(I).* *If two triangles have two angles and the included side of one equal respectively to two angles and the included side of the other, the triangles are congruent* (*a.s.a.* = *a.s.a.*).



Given $\triangle ABC$ with $\angle B = \angle B'$, $\angle C = C'$, and $BC = B'C'$.

To prove $\triangle ABC \cong \triangle A'B'C'$.

Proof

1. Place $\triangle ABC$ on $\triangle A'B'C'$ so that BC shall coincide with its equal $B'C'$.

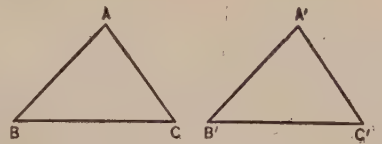
2. BA will fall along $B'A'$.

3. CA will fall along $C'A'$.

4. Point A will fall on point A'.

5. $\therefore \triangle ABC \cong \triangle A'B'C'$.

(I). *If two triangles have two sides and the included angle of one equal respectively to two sides and the included angle of the other, the triangles are congruent* (*s.a.s.* = *s.a.s.*).



Given $\triangle ABC$ and $A'B'C'$ with $AB = A'B'$, $BC = B'C'$, and $\angle B = \angle B'$.

To prove $\triangle ABC \cong \triangle A'B'C'$.

Proof

1. Place $\triangle ABC$ on $\triangle A'B'C'$ so that BC shall coincide with its equal $B'C'$.

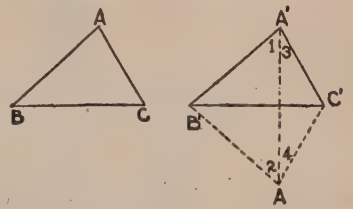
2. BA will fall along $B'A'$.

3. Point A will fall on point A'.

4. AC will coincide with $A'C'$.

5. $\therefore \triangle ABC \cong \triangle A'B'C'$.

(I). *If two triangles have three sides of one respectively equal to three sides of the other, the triangles are congruent* (*s.s.s.* = *s.s.s.*).



Given $\triangle ABC$ and $A'B'C'$ with $AB = A'B'$, $BC = B'C'$, and $AC = A'C'$.

To prove $\triangle ABC \cong \triangle A'B'C'$.

Proof

1. Place $\triangle ABC$ so that BC shall coincide with its equal $B'C'$, and A and A' lie on opposite sides of $B'C'$.

2. Draw AA'. **3.** $\triangle AB'A'$ is isosceles.

4. $\therefore \angle 1 = \angle 2$ **5.** $\triangle AC'A'$ is isosceles.

6. $\therefore \angle 3 = \angle 4$. **7.** $\angle A = \angle A'$.

8. $\therefore \triangle AB'C' \cong \triangle A'B'C'$.

9. *i.e.*, $\triangle ABC \cong \triangle A'B'C'$.

* The Roman numerals preceding each proposition refer to the book in which the proposition is treated in textbooks.

CONGRUENT TRIANGLES

SUMMARY

Triangles can be proved congruent by—

1. *s.a.s.* = *s.a.s.*
2. *a.s.a.* = *a.s.a.*
3. *s.s.s.* = *s.s.s.*
4. *s.a.a.* = *s.a.a.*
5. hy. arm = hy. arm (*right triangles*).
6. hy. acute angle = hy. acute angle (*right triangles*).

EXERCISE 1

1. If at any point in the bisector of an angle, a perpendicular to the bisector is drawn, meeting the sides of the angle, the two triangles thus formed will be congruent.

2. If equal segments, measured from the vertex, are laid off on the sides of an angle, and their extremities joined to any point in the bisector of the angle, two congruent triangles will be formed.

3. If three rods of different lengths are put together, can a triangle always be formed? Can more than one triangle ever be formed?

4. The altitude to the base of an isosceles triangle divides the figure into two congruent triangles.

5. Two triangles are congruent if two sides and the median to one of these sides are equal respectively to two sides and the corresponding median of the other.

6. In a parallelogram $ABCD$ the diagonal AC is trisected by points E and F. Prove triangles AEB and CFD congruent.

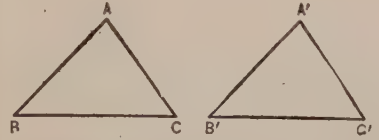
7. If an arm of an isosceles triangle and the line which joins the midpoints of the arms are equal respectively to an arm of another isosceles triangle and the line joining the midpoints of its arms, the two isosceles triangles are congruent.

8. Two isosceles triangles are congruent if the vertex angle and the altitude upon an arm of one triangle are equal respectively to the vertex angle and the corresponding altitude of the other.

9. Two triangles are congruent if a side, an adjacent angle, and the bisector of this angle are equal respectively to a side, an adjacent angle, and the bisector of this angle in the other triangle.

10. Prove that two trapezoids are congruent if their sides, taken in order, are equal each to each.

(I). *If two triangles have two angles and the side opposite one of them equal respectively to two angles and the corresponding side of the other, the triangles are congruent* (*s.a.a.* = *s.a.a.*).



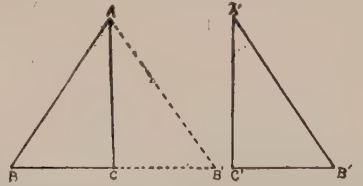
Given $\triangle ABC$ and $A'B'C'$ with $BC = B'C'$, $\angle B = \angle B'$, and $\angle A = \angle A'$.

To prove $\triangle ABC \cong \triangle A'B'C'$.

Proof

1. $\angle B = \angle B'$.
2. $\angle A = \angle A'$.
3. $\therefore \angle C = \angle C'$.
4. $BC = B'C'$.
5. $\therefore \triangle ABC \cong \triangle A'B'C'$.

(I). *If two right triangles have the hypotenuse and an arm of the one respectively equal to the hypotenuse and arm of the other, the triangles are congruent* (*hy. arm* = *hy. arm*).



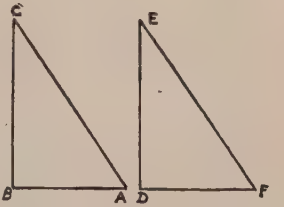
Given $\triangle ABC$ and $A'B'C'$ with $AB = A'B'$, $AC = A'C'$, and the rt. $\measuredangle C$ and C'.

To prove $\triangle ABC \cong \triangle A'B'C'$.

Proof

1. Place $\triangle A'B'C'$ so that $A'C'$ coincides with its equal AC, and points B and B' fall on opposite sides of AC.
2. $\angle BCA$ and $\angle B'CA$ are rt. $\measuredangle$.
3. $\therefore BCB'$ is a st. line.
4. $\therefore BCB'A$ is an isosceles triangle.
5. $\angle B = \angle B'$. 6. $AC = AC$.
7. $\angle BCA = \angle B'CA$.
8. $\triangle ABC \cong \triangle AB'C$.
9. $\therefore \triangle AB'C \cong \triangle A'B'C'$.

(I). *If two right triangles have the hypotenuse and an acute angle of one equal respectively to the hypotenuse and an acute angle of the other, the triangles are congruent.*



Given $\triangle ABC$ and FDE with $AC = FE$, $\angle A = \angle F$, and $\measuredangle B$ and D right angles.

To prove $\triangle ABC \cong \triangle FDE$.

Proof

1. $\angle A = \angle F$.
2. $\angle B = \angle D$.
3. $AC = EF$.
4. $\therefore \triangle ABC \cong \triangle FDE$.

CONGRUENT TRIANGLES—*Concluded*

SUMMARY

Lines or angles can often be proved equal by showing that—

1. Each is equal to some third line or angle.
2. They are corresponding parts of congruent triangles.
   (a) Select two triangles each of which contains one of the parts to be proved equal. Prove these triangles congruent.
   (b) If the figure does not contain such triangles, try to make them by drawing other lines in definite ways, such as joining two points in the figure, bisecting an angle, drawing parallels to lines in the figure, etc.
   (c) If the triangles selected cannot be proved congruent directly, try to prove some other pair of triangles in the figure congruent and then use their equal corresponding parts to prove the first pair of triangles congruent.

EXERCISE 2

1. The lines bisecting the base angles of an isosceles triangle and included within the triangle are equal.

2. If two medians of a triangle are perpendicular to the sides to which they are drawn, the triangle is equilateral.

3. The median to the base of an isosceles triangle bisects the angle at the vertex and is perpendicular to the base.

4. The medians to the arms of an isosceles triangle are equal.

5. Lines from the midpoints of the arms of an isosceles triangle to the midpoint of the base are equal.

6. If the base of an isosceles triangle is trisected and from each point of division perpendiculars are drawn upon the nearest arm, these perpendiculars are equal. State the converse of this proposition.

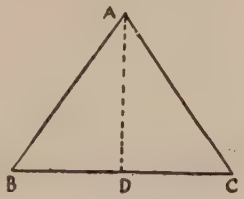
7. In the figure of Ex. 6, prove that the perpendiculars from the trisection points upon the more distant arms are equal.

8. If two equal circles whose centers are O and O' intersect in A and B, prove $\angle OAO' = \angle OBO'$.

9. If two circles (equal or unequal) intersect at A and B, and OO' intersects AB at D, prove $AD = DB$.

10. In triangle ABC, CB is produced its own length to D, and AB is produced its own length to E. DE is drawn. A line is drawn through B, meeting DE in F and AC in G. Prove $FB = BG$.

(I). *In an isosceles triangle, the angles opposite the equal sides are equal.*



Given isos. $\triangle ABC$ with $AB = AC$.

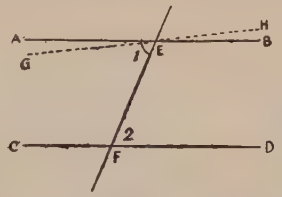
To prove $\angle B = \angle C$.

Proof

1. Draw AD bisecting $\angle A$.
2. $AB = AC$.
3. $AD = AD$.
4. $\angle BAD = \angle CAD$.
5. $\triangle BAD \cong \triangle CAD$.
6. $\therefore \angle B = \angle C$.

Cor. *An equilateral triangle is equiangular.*

(I). *If two parallels are cut by a transversal, the alternate-interior angles are equal.*



Given parallel lines AB and CD, and $\measuredangle 1$ and 2 any pair of alt.-int. $\measuredangle$.

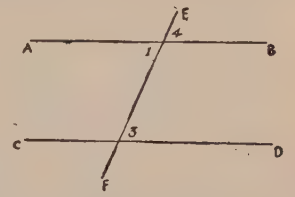
To prove $\angle 1 = \angle 2$.

Proof

1. Either $\angle 1 = \angle 2$, or $\angle 1 \neq \angle 2$.
2. Suppose $\angle 1 \neq \angle 2$. Draw GH making $\angle GEF = \angle 2$.
3. Then GH is $\parallel$ to CD.
4. There would then be two lines through E, both parallel to CD, which is impossible.
5. $\therefore \angle 1 = \angle 2$.

Cor. *If a transversal is perpendicular to one of two parallel lines, it is perpendicular to the other also.*

(I). *If two parallels are cut by a transversal, the corresponding angles are equal.*



Given parallel lines AB and CD, and $\measuredangle 3$ and 4 any pair of corresponding angles.

To prove $\angle 3 = \angle 4$.

Proof

1. $\angle 1 = \angle 4$.
2. $\angle 1 = \angle 3$.
3. $\therefore \angle 3 = \angle 4$.

EQUAL ANGLES

EXERCISE 3

1. Prove the correctness of the following construction for bisecting the angle ABC: On AB produced beyond B, take $BD = BC$, and draw a line through B parallel to CD.

2. A straight line drawn through the midpoint of a diagonal of a parallelogram and terminated by the opposite sides is bisected at that point.

3. ABC is an isosceles triangle with $AB = BC$. Through B lines are drawn, meeting AC in D and E respectively, and making $\angle ABD = \angle EBC$. Prove that $\angle BDE = \angle BED$.

4. The line drawn through the vertex of an isosceles triangle parallel to the base bisects the exterior angle at the vertex.

5. In the equilateral triangle ABC, D, E, and F are points on AC, AB and BC respectively, and $AD = CF = BE$. Prove that triangle DEF is equilateral.

6. If lines are drawn from any point in the bisector of the vertex angle of an isosceles triangle to the ends of the base, these lines make equal angles with the base.

7. If ABC and ADC are two isosceles triangles on the same base, then $\angle BAD = \angle BCD$.

8. The line bisecting the vertex angle of an isosceles triangle also bisects the base.

9. (a) If equal segments measured from the ends of the base are laid off on the base of an isosceles triangle, lines drawn from the vertex of the triangle to the ends of the segments are equal. (*b*) Prove this exercise when the equal segments are laid off on the base produced.

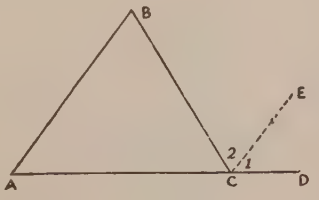
10. If in quadrilateral $ABCD$, $AB = BC$ and $AD = DC$, then $\angle A = \angle C$.

11. The lower base angles of an isosceles trapezoid are equal. Prove this exercise, using as construction lines perpendiculars from the ends of the smaller base upon the larger base.

12. Prove the preceding exercise, using in the figure a line drawn through one end of the smaller base parallel to the opposite non-parallel side of the trapezoid.

13. Prove the converse of the preceding exercise.

(I). *An exterior angle of a triangle is equal to the sum of its two remote interior angles.*



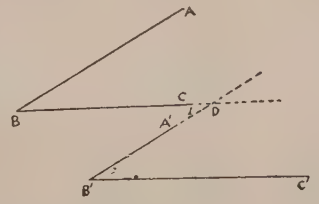
Given $\triangle ABC$ and ext. $\angle BCD$.

To prove $\angle BCD = \angle A + \angle B$.

Proof

1. Draw $CE \parallel AB$.
2. $\angle 2 = \angle B$.
3. $\angle 1 = \angle A$.
4. $\angle BCD = \angle 1 + \angle 2$.
5. $\therefore \angle BCD = \angle A + \angle B$.

(I). *If two angles have their corresponding sides parallel and extending in the same direction from their vertices, they are equal.*



Given $\angle B$ and $\angle B'$ with $AB \parallel A'B'$, and $BC \parallel B'C'$.

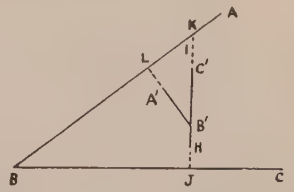
To prove $\angle B = \angle B'$.

Proof

1. Produce, if necessary, BC and $B'A'$ to meet at D.
2. $BA \parallel B'A'$, and $BC \parallel B'C'$.
3. $\therefore \angle B = \angle 1$, and $\angle B' = \angle 1$.
4. $\therefore \angle B = \angle B'$.

Cor. *Two angles whose corresponding sides are parallel are either equal or supplementary.*

(I). *If two angles have their sides respectively perpendicular, they are either equal or supplementary.*



Given $\angle ABC$ and $\angle A'B'C'$, with $A'B' \perp AB$, and $B'C' \perp BC$.

To prove $\angle ABC = \angle A'B'C'$, and $\angle ABC$ sup. of $\angle A'B'H$.

Proof

1. Extend $B'C'$ to K, $B'H$ to J, and $B'A'$ to L.
2. $\angle ABC$ and $\angle 1$ are comp.
3. $\angle A'B'C'$ and $\angle 1$ are comp.
4. $\therefore \angle ABC = \angle A'B'C'$.
5. $\angle A'B'C'$ and $\angle A'B'H$ are sup.
6. $\therefore \angle ABC$ and $\angle A'B'H$ are sup.

EQUAL ANGLES—*Concluded*

EXERCISE 4

1. If at the ends of the base of an isosceles triangle perpendiculars are erected on the arms, another isosceles triangle is formed.

2. The bisector of the angle C in the triangle ABC meets AB in D. DE is drawn parallel to AC, meeting BC in E and the bisector of the exterior angle at C in F. Prove $DE = EF$.

3. ABC and DBC are two triangles on the same base BC and on the same side of it. $AB = DC$, $AC = DB$, and AC intersects DB at O. Prove that triangle BOC is isosceles.

4. If the bisector of an angle of a triangle is perpendicular to the opposite side, the triangle is isosceles.

5. If from the ends of the base BC of isosceles triangle ABC, equal parts BD and CE are laid off on one arm and the prolongation of the other, prove that the line joining D and E is bisected by the base.

(*Hint.*—Through D draw DF parallel to CE, meeting BC at F.)

6. In the triangle ABC, perpendiculars from A and B to AC and BC respectively meet at D, and $AD = BD$. Prove that triangle ABC is isosceles.

7. If from any point in the bisector of an angle, a line is drawn parallel to one side of the angle, an isosceles triangle is formed.

8. On diagonal AC of square $ABCD$, point E is taken so that $AE = AB$. Through E a perpendicular to AE is drawn, cutting BC in F. Prove that $BF = EC$.

9. Perpendiculars from opposite vertices of a parallelogram to the diagonal not containing these points are equal.

10. The diagonals of a rectangle are equal.

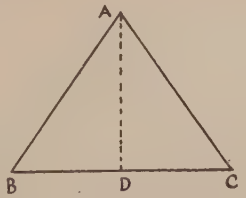
11. If one angle of a parallelogram is one and one-fifth right angles, find each of the remaining angles.

12. If through the vertices of a triangle, lines are drawn parallel to the opposite sides of the triangle, the lines which join the vertices of the triangle thus formed to the opposite vertices of the given triangle are bisected by the sides of the given triangle.

13. Draw a line parallel to the base of a triangle so that the segment of this line intercepted between the sides of the triangle shall equal a given line.

14. The diagonals of an isosceles trapezoid are equal; and conversely.

(I). *If two angles of a triangle are equal, the sides opposite these angles are equal.*



Given $\triangle ABC$ with $\angle B = \angle C$.

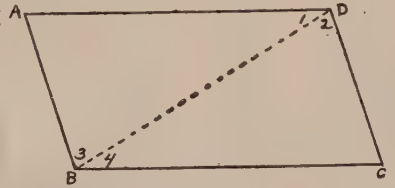
To prove $AB = AC$.

Proof

1. Draw AD bisecting $\angle BAC$.
2. $\angle B = \angle C$.
3. $\angle BAD = \angle DAC$.
4. $AD = AD$.
5. $\triangle ACD \cong \triangle BAD$.
6. $\therefore AB = AC$.

Cor. *An equiangular triangle is also equilateral.*

(I). *The opposite sides and the opposite angles of a parallelogram are equal.*



Given $\square ABCD$.

To prove $AB = DC$, $BC = AD$. $\angle A = \angle C$, and $\angle B = \angle D$.

Proof

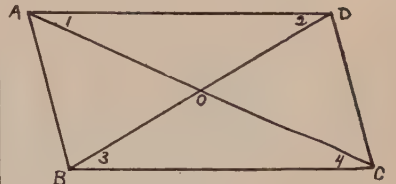
1. Draw BD. **2.** $AD \parallel BC$, $AB \parallel DC$
3. $\angle 1 = \angle 4$, $\angle 3 = \angle 2$.
4. $BD = BD$.
5. $\triangle ABD \cong \triangle BCD$.
6. $\therefore AB = DC$, $BC = AD$, $\angle A = \angle C$.
7. In the same way, by drawing AC, we can prove $\angle B = \angle D$.

Cor. 1. *A diagonal divides a parallelogram into two congruent triangles.*

Cor. 2. *Parallels included between parallels are equal.*

Cor. 3. *Parallel lines are everywhere equidistant.*

(I). *The diagonals of a parallelogram bisect each other.*



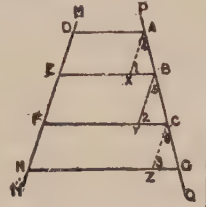
Given $\square ABCD$ with diagonals BD and AC.

To prove $BO = OD$, and $CO = OA$.

Proof

1. $AD \parallel BC$.
2. $\angle 1 = \angle 4$, $\angle 2 = \angle 3$.
3. $AD = BC$.
4. $\triangle AOD \cong \triangle BOC$.
5. $\therefore BO = OD$, and $CO = OA$.

(I). *If three or more parallels intercept equal segments on one transversal, they intercept equal segments on every transversal.*



Given
$AD \| BE \| CF \| GH$
and
$DE = EF = FH$.

To prove
$AB = BC = CG$.

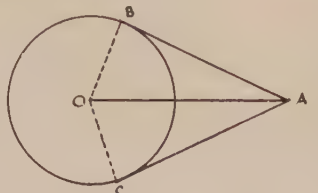
Proof

1. Draw AX, BY, and $CZ \| MN$.
2. $AX \| BY \| CZ$.
3. $AX = DE$, and $BY = EF$.
4. But $DE = EF$.
5. $\therefore AX = BY$.
6. $\angle ABX = \angle BCY$, and $\angle 4 = \angle 5$.
7. $\therefore \triangle ABX \cong \triangle BCY$.
8. Hence, $AB = BC$.
9. Similarly, by using $\triangle BCY$ and CGZ, we can prove that $BC = CG$.
10. $\therefore AB = BC = CG$.

Cor. 1. *The line parallel to one side of a triangle, bisecting another side, bisects the third side.*

Cor. 2. *The line bisecting one non-parallel side of a trapezoid and parallel to the bases bisects the other non-parallel side.*

(II). *The tangents drawn to a circle from a point without are equal and make equal angles with the line joining the point to the center of the circle.*



Given AB and AC tangent to $\odot O$, and line AO.

To prove $AB = AC$, and $\angle BAO = \angle CAO$.

Proof

1. Draw OB and OC.
2. $AB \perp OB$, and $AC \perp OC$.
3. $\therefore \triangle OBA$ and OCA are rt. $\triangle$.
4. $OB = OC$.
5. $OA = OA$.
6. $\therefore$ rt. $\triangle AOB \cong$ rt. $\triangle AOC$.
7. $\therefore AB = AC$, and $\angle BAO = \angle CAO$.

EXERCISE 5

1. A line drawn perpendicular to either leg of a right triangle at its middle point passes through the middle point of the hypotenuse.

2. In any parallelogram $ABCD$, the diagonal AC is trisected by the line drawn from B to the midpoint of CD and the line drawn from D to the midpoint of AB.

3. A line joining the midpoints of two sides of a triangle bisects a line drawn from any point in the third side to the opposite vertex.

4. The common internal tangents of two circles are equal.

5. The common external tangents of two circles are equal.

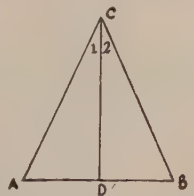
6. Two circles are tangent at T. Line ATB is drawn through T, terminating in the two circles. Prove that the tangents at A and B are parallel.

7. Tangents to a circle at the extremities of any two diameters form a rhombus.

8. If two circles are tangent, prove that the tangents to them from any point in their common internal tangent are equal.

9. If two circles are tangent externally, their common internal tangent bisects their common external tangents.

(I). *The bisector of the vertex angle of an isosceles triangle is perpendicular to the base and bisects the base.*



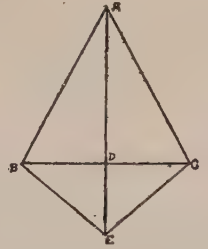
Given $\triangle ABC$ with $AC = BC$, and $\angle 1 = \angle 2$.

To prove $CD \perp AB$, and CD bisects AB.

Proof

1. $CA = CB$.
2. $CD = CD$.
3. $\angle 1 = \angle 2$.
4. $\therefore \triangle ADC \cong \triangle BDC$.
5. $AD = DB$.
6. $\angle ADC = \angle BDC$.
7. $\therefore \angle ADC = \frac{1}{2} \angle ADB$.
8. But $\angle ADB$ is a st. $\angle$.
9. $\therefore \angle ADC$ is a rt. $\angle$.
10. $\therefore CD \perp AB$, and CD bisects AB.

(I). *Two points, each equidistant from the ends of a line, determine the perpendicular bisector of the line.*



Given line BC and points A and E such that $AB = AC$ and $BE = EC$.

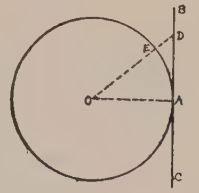
To prove AE is the perpendicular bisector of BC.

Proof

1. $AB = AC$, and $BE = EC$.
2. $AE = AE$.
3. $\triangle ABE \cong \triangle ACE$.
4. $\angle BAE = \angle CAE$.
5. $\triangle ABC$ is isosceles.
6. $\therefore AE$ is the perpendicular bisector of BC.

Cor. *Every point equidistant from the ends of a line lies in the perpendicular bisector of the line.*

(II). *A tangent is perpendicular to the radius drawn to the point of contact.*



Given BC tangent to $\odot O$ at A, and radius OA.

To prove $BC \perp OA$.

Proof

1. Let D be any point in BC except A. Draw OD.
2. D is outside $\odot O$.
3. $\therefore OD > OE$.
4. $OE = OA$.
5. $\therefore OD > OA$.
6. This shows that OA is the shortest line from O to BC.
7. $\therefore OA \perp BC$; *i.e.*, $BC \perp OA$.

Cor. *At a given point on a circle, there can be one tangent only.*

PERPENDICULAR LINES

SUMMARY

To prove lines perpendicular, use one of the following:

1. An angle equal to its supplement is a right angle.
2. An angle inscribed in a semicircle is a right angle.
3. If the sum of two angles of a triangle is a right angle, the third angle is a right angle.
4. An angle equal to an angle already known to be a right angle is a right angle.
5. A tangent is perpendicular to the radius drawn to the point of contact.
6. Two points, each equally distant from the ends of a line, determine the perpendicular bisector of the line.
7. The shortest line from a point outside of a given line to the line is the perpendicular from the point to the line.
8. Half a straight angle is a right angle.

EXERCISE 6

1. The bisectors of two supplementary-adjacent angles are perpendicular to each other.

2. If two adjacent sides of a quadrilateral are equal, and the other two sides are also equal, the diagonals of the quadrilateral intersect at right angles.

3. If one leg of an isosceles triangle is the diameter of a circle, the circle bisects the base.

4. The circle described on one of the sides of an equilateral triangle as diameter bisects the other two sides.

5. From an external point P a tangent PT is drawn to a circle whose center is O. T' is a point on the circle such that $PT' = PT$. Prove that PT' is perpendicular to OT'.

6. The diagonals of a rhombus intersect at right angles.

7. Two angles of a triangle are to each other as $7:2$, and their difference is $50°$. Find the number of degrees in the third angle.

8. The median to the base of an isosceles triangle is perpendicular to the base.

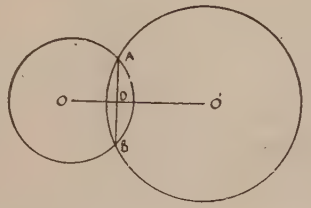
9. If two isosceles triangles have the same base, the line joining their vertices is the perpendicular bisector of the common base.

10. In triangle ABC, $AC = BC$. BF and AE are drawn, meeting AC and BC respectively in F and E, and intersecting each other in D. If $\angle DBA = \angle DAB$, prove that CD is perpendicular to AB.

11. Two circles touch externally at C, and a tangent AB is drawn touching the circles at A and B respectively. Prove that $\angle ACB$ is a right angle.

12. If one of the legs of an isosceles triangle is produced through the vertex by its own length, the line joining the end of the leg produced to the nearer end of the base is perpendicular to the base.

(II). *If two circles intersect, the line of centers bisects the common chord at right angles.*



Given ⓢ O and O', intersecting at A and B.

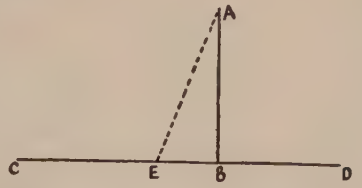
To prove OO' is the perpendicular bisector of AB.

Proof

1. O and O' are each equally distant from A and B.

2. $\therefore OO'$ is the perpendicular-bisector of AB.

(I). *From a point without a line, there can be only one perpendicular to that line.*



Given point A outside line CD, and $AB \perp CD$.

To prove AB is the only perpendicular that can be drawn from A to CD.

Proof

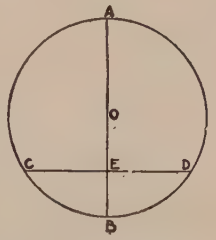
1. If possible, let AE be another perpendicular from A to CD. Then AEB is a rt. $\angle$.

2. ABC is a rt. $\angle$.

3. We would then have two rt. $\measuredangle$ in $\triangle ABE$, which is impossible.

4. $\therefore AE$ cannot be $\perp CD$, and AB is the only perpendicular from A to CD.

(II). *If a diameter bisects a chord, which is not itself a diameter, it is perpendicular to the chord.*



Given $\odot O$, and diameter AB bisecting chord CD.

To prove $AB \perp CD$.

Proof

1. O is equidistant from C and D.

2. E is equidistant from C and D.

3. AB passes through O and E.

4. $\therefore AB$ is $\perp CD$.

PERPENDICULAR LINES—*Concluded*

EXERCISE 7

1. If a diameter bisects a chord which is not itself a diameter, it bisects the two arcs of the chord.

2. If two circles intersect, the line joining the center of one circle to the midpoint of the common chord, passes through the center of the other circle.

3. If the line joining the centers of two intersecting circles is bisected by the common chord, the circles are equal.

4. The line joining the midpoints of two parallel chords is a diameter.

5. The chord of one-half an arc is greater than one-half the chord of the arc.

6. Two circles with centers O and O' intersect at A and B. Prove that the center of any circle passing through A and B lies on the line OO' or its prolongation.

7. The line joining the center of a circle with the midpoint of a chord bisects any chord parallel to the given chord.

8. If one angle of a parallelogram is a right angle, its other angles are also right angles.

9. AB and AC are lines drawn from point A on a circle to the extremities of a diameter BC. $\angle ABC$ is $2\frac{1}{2}$ times as large as $\angle ACB$. Find the number of degrees in each of these angles.

10. The arms BA and CA of isosceles triangle ABC are produced through the vertex A to points D and E respectively, so that $AD = AE$. Prove that the line joining the midpoint of DE to point A is perpendicular to BC.

11. *Prove:* Circles described on any two sides of a triangle as diameters intersect on the third side or the third side produced.

12. (*a*) Given two points A and B. Find a point E so that $\angle AEB$ shall be a right angle. (*b*) Given points A, B, C, and D. Find a point E such that both $\angle AEB$ and $\angle CED$ shall be right angles. Is this problem always possible?

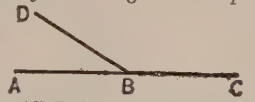
13. If a diagonal of a quadrilateral bisects two of its angles, it is perpendicular to the other diagonal.

14. If two consecutive angles of a quadrilateral are right angles, the bisectors of the other two angles are perpendicular to each other.

15. Tangents drawn to a circle at opposite ends of a diameter are parallel.

X—SUPPLEMENTARY ANGLES

(I). *If two adjacent angles have their exterior sides in a straight line, they are supplementary; and conversely, if two adjacent angles are supplementary, their exterior sides are in a straight line.*



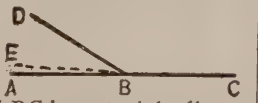
Given adj. $\measuredangle DBA$ and DBC, and st. line ABC.

To prove $\measuredangle DBA$ and DBC are supplementary.

Proof

1. $\angle DBA + \angle DBC = \angle ABC$.
2. But $\angle ABC$ is a st. $\angle$.
3. $\therefore \measuredangle DBA$ and DBC are supplementary.

Converse



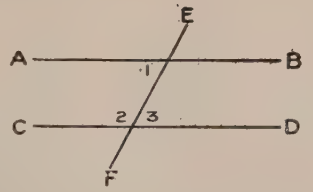
Given sup. $\measuredangle ABD$ and CBD.

To prove ABC is a straight line.

Proof

1. Either AB and BC form a straight line or they do not.
2. Suppose that they do not form a straight line, and that EBC is a st. line.
3. Then $\angle DBE$ is sup. of $\angle CBD$.
4. But $\angle ABD$ is sup. of $\angle CBD$.
5. $\therefore \angle DBE = \angle ABD$.
6. This is impossible.
7. $\therefore ABC$ is a st. line.

(I). *If two parallels are cut by a transversal, the interior angles on the same side of the transversal are supplementary.*



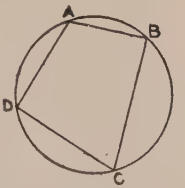
Given $AB \parallel CD$, and the int. $\measuredangle$ 1 and 2 lying on the same side of a transversal EF.

To prove $\angle 1$ is the sup. of $\angle 2$.

Proof

1. $AB \parallel CD$.
2. $\angle 1 = \angle 3$.
3. $\angle 3$ is sup. of $\angle 2$.
4. $\therefore \angle 1$ is sup. of $\angle 2$.

(II). *The opposite angles of an inscribed quadrilateral are supplementary.*



Given inscribed quad. $ABCD$.

To prove $\angle A + \angle C = 180°$ and $\angle B + \angle D = 180°$.

Proof

1. $\angle A$ is meas. by $\frac{1}{2}\overset{\frown}{BCD}$.
2. $\angle C$ is meas. by $\frac{1}{2}\overset{\frown}{BAD}$.
3. $\therefore \angle A + \angle C$ is meas. by $\frac{1}{2}\overset{\frown}{BCD} + \frac{1}{2}\overset{\frown}{BAD}$, or $\frac{1}{2}(\overset{\frown}{BCD} + \overset{\frown}{BAD})$.
4. But $\overset{\frown}{BCD} + \overset{\frown}{BAD} = 360°$
5. $\therefore \frac{1}{2}(\overset{\frown}{BCD} + \overset{\frown}{BAD}) = 180°$.
6. *i.e.* $\angle A + \angle C = 180°$.
7. In the same manner, we can prove that $\angle B + \angle D = 180°$.

SUPPLEMENTARY AND COMPLEMENTARY ANGLES

SUMMARY

1. Two adjacent angles whose exterior sides lie in the same straight line are supplementary.

2. Interior angles on the same side of a transversal cutting parallel lines are supplementary.

3. Opposite angles of an inscribed quadrilateral are supplementary.

4. The acute angles of a right triangle are complementary.

5. The sum of the angles of a triangle is equal to two right angles.

EXERCISE 8

1. If two parallel lines are cut by a transversal, the exterior angles on the same side of the transversal are supplementary.

2. If the lower base angles of a trapezoid are equal, the upper base angles are also equal.

3. Two circles intersect at A and B. Through A and B lines are drawn, meeting the circles at C and D, and F and G respectively. Prove that angles ADG and FCA are supplementary.

4. Any angle of an inscribed quadrilateral is equal to the opposite exterior angle.

5. If a perpendicular is erected at any point in the base of an isosceles triangle, meeting one leg and the other leg produced, another isosceles triangle is formed external to the given one.

6. The bisectors of two consecutive angles of a parallelogram are perpendicular to each other.

7. AB and CD are parallel tangents, meeting a third tangent at A and C respectively. O is the center of the circle. Prove that $\angle AOC$ is a right angle.

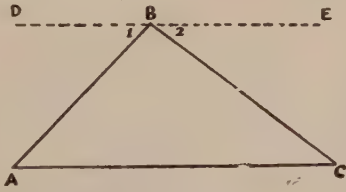
8. The bisectors of two vertical angles form a straight line.

9. If from one point of intersection of two circles two diameters are drawn, the other extremities of the diameters and the other point of intersection of the circles lie in the same straight line.

10. Two straight lines meet, and one of the angles formed is six times as great as its adjacent angle. How many degrees are there in each angle?

11. How many degrees are there in an angle whose complement is equal to one-fourth of its supplement?

(I). *The sum of the angles of a triangle is equal to two right angles.*



Given $\triangle ABC$.

To prove $\angle A+\angle ABC+\angle C=$ 2 rt. $\measuredangle$.

Proof

1. Through B, draw $DE \| AC$.

2. $\angle 1=\angle A$ and $\angle 2=\angle C$.

3. $\angle 1+\angle ABC+\angle 2=2$ rt. $\measuredangle$.

4. $\therefore \angle A+\angle ABC+\angle C=2$ rt. $\measuredangle$

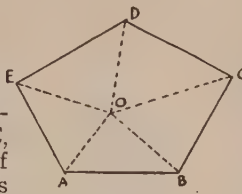
Cor. 1. *In a triangle there can be at most one obtuse angle or one right angle.*

Cor. 2. *The acute angles of a right triangle are complementary.*

Cor. 3. *If two triangles have two angles of the one respectively equal to two angles of the other, the third angles are equal.*

Cor. 4. *Each angle of an equiangular triangle is equal to* 60°.

(I). *The sum of the angles of a convex polygon of n sides is* $(2n-4)$ *right angles.*



Given polygon $ABCDE$, the number of whose sides is represented by n.

To prove the sum of the angles of the polygon is equal to $(2n-4)$ right angles.

Proof

1. Let O be any point within the polygon. Draw OA, OB, OC, etc.

2. n $\triangle$ are thus formed with vertices at O.

3. The sum of all $\measuredangle$ in the $\triangle$ is $2n$ rt. $\measuredangle$.

4. The sum of the vertex $\measuredangle$ of the $\triangle$ is 4 rt. $\measuredangle$.

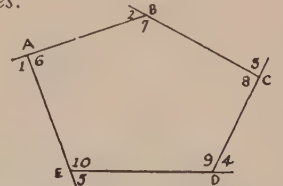
5. $\therefore$ the sum of the base $\measuredangle$ of the $\triangle$ is $(2n-4)$ rt. $\measuredangle$.

6. But the sum of the $\measuredangle$ of the polygon = the sum of the base $\measuredangle$ of the $\triangle$.

7. $\therefore$ the sum of the $\measuredangle$ of the polygon $=(2n-4)$ rt. $\measuredangle$.

Cor. *Each angle of an equiangular polygon of n sides is equal to* $\frac{2n-4}{n}$ rt. $\measuredangle$.

(I). *If the sides of any polygon be produced in succession, forming one exterior angle at each vertex, the sum of the angles is equal to four right angles.*



Given polygon $ABCDE$, the number of whose sides is represented by n, and the exterior $\measuredangle$1, 2, 3, etc.

To prove the sum of the ext. $\measuredangle$ = 4 rt. $\measuredangle$.

Proof

1. $\angle 1+\angle 6=2$ rt. $\measuredangle$; $\angle 2+\angle 7=$ 2 rt. $\measuredangle$; $\angle 3+\angle 8=2$ rt. $\measuredangle$; etc.

i.e., the sum of the int. and ext. $\measuredangle$ at each vertex = 2 rt. $\measuredangle$.

2. Hence, if I represents the sum of all the int. $\measuredangle$, and E the sum of all the ext. $\measuredangle$, $I+E=2n$ rt. $\measuredangle$.

3. But $I=(2n-4)$ rt. $\measuredangle$.

4. $\therefore E=2n-(2n-4)$ rt. $\measuredangle=4$ rt. $\measuredangle$.

i.e., the sum of the ext. $\measuredangle=4$ rt. $\measuredangle$.

Cor. *Each exterior angle of an equiangular polygon of n sides is equal to* $\frac{360°}{n}$.

SUM OF ANGLES OF POLYGONS

EXERCISE 9

1. The angles of a triangle are in the ratio 4:3:2. Find the number of degrees in each angle.

2. The vertex angle of an isosceles triangle is 18° 30′. Find the number of degrees in each base angle. Check your answer.

3. Prove that the sum of the angles of any quadrilateral is 360°.

4. Two angles of a triangle are 75° and 40° respectively. Find the number of degrees in the angles formed by the bisectors of the angles of the triangle. Check your answer.

5. The angle formed by two tangents is the supplement of the angle formed by the radii drawn to the points of contact.

6. If two opposite sides of a quadrilateral are produced, the sum of the two exterior angles thus formed is equal to the sum of the two non-adjacent interior angles.

7. If the altitudes of triangle ABC meet in O, the angles BOC and BAC are supplementary.

8. An angle of a regular polygon and the central angle of the same polygon are supplementary.

9. An exterior angle at the base of an isosceles triangle is equal to the angle included between the bisectors of the interior base angles.

10. Find the number of sides of an equiangular polygon if (a) an interior angle is 135°; (b) an exterior angle is 72°.

11. How many sides has the polygon the sum of whose interior angles is twice the sum of the exterior angles formed by producing each of its sides in succession?

12. Is it possible for a regular polygon to have an exterior angle of 7°? Of 5°? How many sides has each polygon if such exist?

13. In the following table, substitute for each question mark the proper number:

No. sides	Sum of angles	Value of each angle when polygon is equiangular	Value of each exterior angle when polygon is equiangular
n	?	?	?
5	?	?	?
6	?	?	?
8	?	?	?
10	?	?	?
?	?	150°	?
?	?	?	18°
?	900°	?	?
?	?	175°	?

SUM OF TWO LINES OR ANGLES

SUMMARY

To prove the sum of two lines equal to a third line—

1. Lay off one of the two lines on the third line and prove the remainder of the third line equal to the second line.
2. Construct a line in the figure equal to the sum of the first two lines (usually by prolonging one of them), and prove it equal to the third line.

Note.—Similar methods can often be used for angles.

EXERCISE 10

1. The sum of the perpendiculars drawn from any point in the base of an isosceles triangle to the legs is equal to the altitude on either of the legs.

2. If from a point on the prolonged base of an isosceles triangle, perpendiculars are drawn to the two equal sides, their difference is equal to a perpendicular drawn from either extremity of the base to the opposite side.

3. Tangents AC and BD, drawn at the extremities of a diameter AB, meet a third tangent CD at C and D respectively. Prove that $CD = CA + DB$.

4. In triangle ABC, the bisectors of angles A and C meet in O. Through O, DF is drawn parallel to AC, meeting AB and BC in D and E respectively. Prove that $DE = DA + EC$.

5. D is any point in the base BC of isosceles triangle ABC. The side AC is prolonged through C to E so that $CE = CD$. DE is drawn, meeting AB at F. Prove $\angle EFA = 3 \angle AEF$.

6. P is any point in the base BC of isosceles triangle ABC. Through P parallels to the arms of the triangle are drawn, meeting the arms in points E and F respectively. Prove that $PE + PF = CA$.

7. The sum of the perpendiculars from any point within an equilateral triangle to the sides is equal to the altitude of the triangle. (See Ex. 1 above.)

8. AB and AC are tangents drawn to a circle from external point A. D is any point on the minor arc BC. Prove that $\angle ABC = \angle ABD + \angle ACD$.

9. On side AC of $\triangle ABC$ points E and G are taken so that $AE = CG$. ED and GF are drawn parallel to BC, meeting AB in D and F respectively. Prove that $ED + GF = BC$.

10. Side AC of $\triangle ABC$ is divided into five equal parts. Through the points of division, lines are drawn parallel to BC and terminating in AB. Prove that the sum of these lines is equal to $2\ BC$. (*Hint.*—See Ex. 9.)

DOUBLES OF LINES OR ANGLES

SUMMARY

To prove a line or angle double another line or angle—

1. Double the smaller, and prove the result equal to the greater.
2. Bisect the greater, and prove that half is equal to the smaller.

EXERCISE 11

1. The median to the hypotenuse of a right triangle is one-half the hypotenuse.

2. If two medians of a triangle intersect, the larger segment of either median is twice the smaller segment of the same median.

3. The radius of the circumscribed circle of an equilateral triangle is twice the radius of the inscribed circle.

4. The angle formed by two tangents to a circle is twice as great as the angle formed by the radius drawn to a point of contact and the chord joining the points of contact.

5. The angle formed by the bisectors of two angles of an equilateral triangle is double the third angle.

6. If the lower base DC of trapezoid $ABCD$ is double the upper base AB, and the diagonals intersect at O, prove $DO = 2\ OB$, and $CO = 2\ OA$.

7. An exterior angle at the vertex of an isosceles triangle is double the base angle.

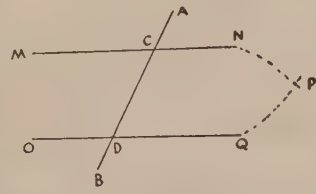
8. An exterior angle at the base of an isosceles triangle diminished by a right angle is equal to one-half the vertex angle.

9. In $\triangle ABC$, AE and CE are the bisectors of $\angle A$ and the exterior angle at C respectively. Prove that $\angle E = \frac{1}{2}\ \angle ABC$.

10. At one extremity of the base of an isoceles triangle a line is drawn perpendicular to the adjacent arm of the triangle. Prove that the angle made by this line and the base produced is equal to one-half the vertex angle.

11. In $\triangle ABC$, the perpendicular bisectors of sides AB and AC meet in point O. Prove that $\angle BOC = 2\ \angle ABC$. (*Hint.*—Point O is the center of the circumscribed circle.)

(I). *If a transversal to two lines makes a pair of alternate interior angles equal, the lines are parallel.*



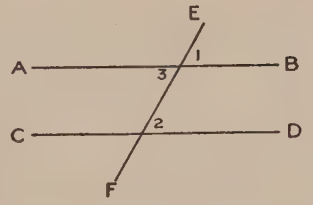
Given lines MN and OQ intersected by AB so that $\angle MCD = \angle CDQ$.

To prove $MN \| OQ$.

Proof

1. MN and OQ are either $\|$ or not $\|$.
2. If MN is not $\| OQ$, these lines will meet at some point P, forming $\triangle DCP$.
3. $\angle MCD$ would then be, greater than $\angle CDQ$.
4. This is impossible.
5. $\therefore$ the supposition that MN is not $\| OQ$ is false.
6. $\therefore MN \| OQ$.

(I). *If a transversal to two lines makes a pair of corresponding angles equal, the lines are parallel.*



Given lines AB and CD, intersected by EF so that $\angle 1 = \angle 2$.

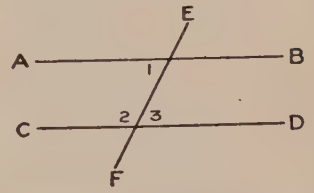
To prove $AB \| CD$.

Proof

1. $\angle 1 = \angle 2$.
2. $\angle 1 = \angle 3$.
3. $\therefore \angle 2 = \angle 3$.
4. $\therefore AB \| CD$.

Cor. *Two lines perpendicular to a third line are parallel.*

(I). *If a transversal to two lines makes a pair of interior angles on the same side of the transversal supplementary, the lines are parallel.*



Given lines AB and CD intersected by EF so that $\angle 1$ and $\angle 2$ are supplementary.

To prove $AB \| CD$.

Proof

1. $\angle 1$ is sup. of $\angle 2$.
2. $\angle 3$ is sup. of $\angle 2$.
3. $\therefore \angle 1 = \angle 3$.
4. $\therefore AB \| CD$.

PARALLEL LINES

SUMMARY

1. **Lines are parallel if—**
   (a) The alternate interior angles are equal.
   (b) The corresponding angles are equal.
   (c) The interior angles on the same side of the transversal are supplementary.
   (d) They are perpendicular or parallel to the same line.
2. A line joining the midpoints of two sides of a triangle is parallel to the third side.
3. The median of a trapezoid is parallel to the bases.
4. A line dividing two sides of a triangle proportionally is parallel to the third side.
5. Opposite sides of a parallelogram are parallel.

EXERCISE 12

1. If two sides of a triangle are produced by their own lengths through a common vertex, the line joining their ends is parallel to the third side.

2. The bisectors of a pair of corresponding angles formed by parallel lines are parallel.

3. The bisectors of the opposite angles of a parallelogram (not a rhombus) are parallel.

4. The bisector of the exterior angle at the vertex of an isosceles triangle is parallel to the base.

5. Prove the converse of Ex. 4.

6. If the sides of one angle are perpendicular, respectively, to the sides of another angle, the line bisecting the first angle is either parallel or perpendicular to the line bisecting the second angle.

7. If the opposite angles of a quadrilateral are equal, the opposite sides are parallel.

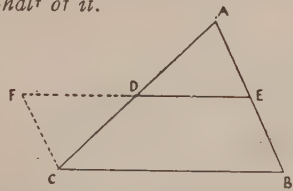
8. Two lines are parallel if a transversal to these lines makes a pair of alternate-exterior angles equal.

9. Two lines are parallel if a transversal to these lines makes a pair of exterior angles on the same side of the transversal supplementary.

10. If through any point P in the bisector of $\angle BAC$, a line is drawn cutting a side of the angle in D, so that $PD = AD$, then PD is parallel to the other side of the angle.

11. Perpendiculars erected at the midpoints of the arms of an isosceles triangle meet and form another isosceles triangle with the line joining the midpoints.

(I). *The line which joins the mid-points of two sides of a triangle is parallel to the third side and equal to one-half of it.*



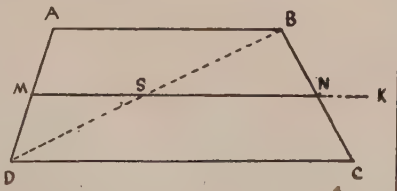
Given $\triangle ABC$ with $AD=DC$ and $AE=EB$.

To prove $DE \| CB$, and $DE=\frac{1}{2}CB$.

Proof

1. Extend ED to F, making $DF=ED$, and draw CF.
2. $AD=CD$. **3.** $DF=DE$.
4. $\angle ADE=\angle CDF$.
5. $\therefore \triangle DFC \cong \triangle ADE$.
6. $CF=AE$. **7.** But $AE=EB$.
8. $\therefore CF=EB$.
9. $\angle DCF=\angle DAE$.
10. $\therefore AB \| FC$. **11.** $FEBC$ is a $\square$.
12. $\therefore FE$ (or DE) $\| CB$.
13. Also $FE=BC$.
14. $DE=\frac{1}{2}FE$. **15.** $\therefore DE=\frac{1}{2}CB$.

(I). *The median of a trapezoid is parallel to the bases and equal to one-half their sum.*



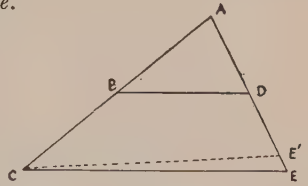
Given trapezoid $ABCD$ with median MN.

To prove $MN \| AB$ and DC, and $MN=\frac{1}{2}(AB+DC)$.

Proof

1. Draw $MK \| AB$.
2. Then $MK \| CD$.
3. MK passes through N.
4. MN coincides with MK.
5. $\therefore MN \| AB$ and DC.
6. Draw DB.
7. S is the mid-point of DB.
8. $\therefore MS=\frac{1}{2}AB$ and $SN=\frac{1}{2}DC$.
9. $\therefore MN=\frac{1}{2}(AB+DC)$.

(III). *If a line divides two sides of a triangle into segments that are proportional, it is parallel to the third side.*



Given $\triangle AEC$, and $AB : BC = AD : DE$.

To prove $BD \| CE$.

Proof

1. Draw $CE' \| BD$, meeting AE in E'.
2. Then $AB : BC = AD : DE'$.
3. But $AB : BC = AD : DE$.
4. $\therefore DE=DE'$.
5. *i.e.*, E and E' coincide.
6. And CE and CE' coincide.
7. $\therefore BD \| CE$.

Cor. *If a line divides two sides of a triangle so that one side is to one of its segments as the other side is to its own corresponding segment, the line is parallel to the third side.*

PARALLEL LINES—*Concluded*

EXERCISE 13

1. A line joining the midpoints of two adjacent sides of any quadrilateral is equal and parallel to the line joining the midpoints of the other two sides.

2. The altitudes to the equal sides of an isosceles triangle meet the sides in A and B. Prove that AB is parallel to the base of the triangle.

3. The bisectors of the equal angles of an isosceles triangle meet the arms at A and B. Prove that AB is parallel to the base of the triangle.

4. From the vertex A of isosceles triangle ABC, equal segments AD and AE are laid off on the arms AB and AC, respectively. F and G are the midpoints of DB and EC, respectively. Prove that FG is parallel to BC.

5. The sum of the perpendiculars dropped from the middle points of two sides of a triangle upon the third side is equal to the altitude on the third side.

6. If from each vertex of a parallelogram a perpendicular is drawn to any straight line lying outside the parallelogram, the sum of the perpendiculars from two opposite vertices is equal to the sum of the perpendiculars from the other two opposite vertices.

7. The sum of the lines drawn from the extremities of any diameter of a circle perpendicular to a tangent is equal to a diameter of the circle.

8. In trapezoid $ABCD$, parallel sides AD and BC are perpendicular to CD. E is the midpoint of AB. Prove $ED = EC$.

9. In the triangle ABC, AD and BE are medians intersecting in O. F is the midpoint of AO and G is the midpoint of BO. Prove that $FGDE$ is a parallelogram.

10. The lines joining the midpoints of the opposite sides of a quadrilateral bisect each other.

11. The sum of the lines joining the midpoints of the adjacent sides of a pentagon is equal to one half the sum of the diagonals of the pentagon.

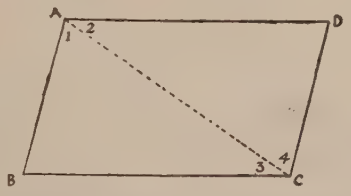
12. The medians of triangle ABC meet at O. D and E are the trisection points of AB. Prove that OD and OE are parallel, respectively, to AC and BC.

13. Two diameters AB and CD are drawn in a circle. Prove that AC is parallel to BD.

14. *Prove:* Each side of a regular hexagon is parallel to another side of the hexagon.

15. The line joining the midpoints of two opposite sides of a parallelogram is parallel to the other two sides.

(I). *If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*



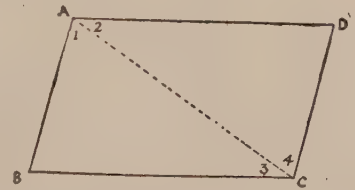
Given quadrilateral $ABCD$ with $AB=DC$, and $AD=BC$.

To prove $ABCD$ is a $\square$.

Proof

1. Draw AC.
2. $AB=DC$, $AD=BC$.
3. $AC=AC$.
4. $\triangle ABC\cong\triangle ACD$.
5. $\angle 1=\angle 4$, and $\angle 3=\angle 2$.
6. $AB\parallel DC$, and $BC\parallel AD$.
7. $\therefore ABCD$ is a $\square$.

(I). *If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.*



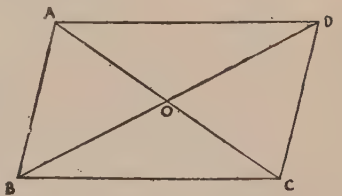
Given quadrilateral $ABCD$ with $AD\parallel BC$ and $AD=BC$.

To prove $ABCD$ is a $\square$.

Proof

1. Draw AC.
2. $AD\parallel BC$.
3. $\therefore \angle 2=\angle 3$.
4. $AD=BC$.
5. $AC=AC$.
6. $\triangle ADC\cong\triangle ABC$.
7. $\angle 4=\angle 1$.
8. $AB\parallel DC$.
9. $\therefore ABCD$ is a $\square$.

(I). *If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.*



Given quadrilateral $ABCD$ with $AO=OC$ and $BO=OD$.

To prove $ABCD$ is a $\square$.

Proof

1. $AO=OC$, $BO=OD$.
2. $\angle AOD=\angle BOC$.
3. $\triangle AOD\cong\triangle BOC$.
4. $\angle OAD=\angle OCB$.
5. $BC\parallel AD$.
6. In the same way, by using $\triangle AOB$ and COD, we can prove $AB\parallel DC$.
7. $\therefore ABCD$ is a $\square$.

PARALLELOGRAMS

SUMMARY

Quadrilaterals Classified

- **Quadrilateral** (*Four sides*)
  - 1. **Parallelogram** (Opposite sides parallel)
    - **Rectangle** (Right-angled)
      - **Square** (Equilateral)
    - **Rhombus** (Equilateral)
      - **Square** (Right-angled)
  - 2. **Trapezoid** (Two sides parallel)
    - **Isosceles** (Non-parallel sides equal)
  - 3. **Trapezium** (No sides parallel)

To prove a quadrilateral a parallelogram, prove one of the following:

1. The opposite sides are parallel (*both pairs*).
2. The opposite sides are equal (*both pairs*).
3. Two opposite sides are both equal and parallel.
4. The diagonals bisect each other.

To prove a parallelogram a rectangle, prove one of the following:

1. One angle is a right angle. (See Ex. 1, below.)
2. The diagonals are equal. (See Ex. 3, below.)

To prove a parallelogram a square, prove that two adjacent sides are equal and one angle is a right angle. (See Exs. 1 and 2, below.)

To prove a parallelogram a rhombus, prove that two adjacent sides are equal. (See Ex. 2, below.)

EXERCISE 14

1. If one angle of a parallelogram is a right angle, each angle is a right angle, and the figure is a rectangle.

2. If two adjacent sides of a parallelogram are equal, the figure is equilateral and therefore a rhombus.

3. If the diagonals of a parallelogram are equal, the figure is a rectangle; and conversely.

4. The diagonals of a rhombus are perpendicular to each other.

5. If the midpoints of the adjacent sides of any quadrilateral are joined in order, a parallelogram is formed.

6. If the midpoints of each half of the diagonals of a parallelogram are joined, a parallelogram is formed.

7. If lines are drawn in a circle connecting the ends of two diameters, a rectangle is formed.

8. If the diagonals of a parallelogram are perpendicular to each other, the figure is a rhombus.

9. $ABCD$ is a parallelogram. On the diagonal BD, produced at both ends, points E and F are taken so that $BE = DF$. Prove that $AECF$ is a parallelogram.

10. The bisectors of the interior angles of a parallelogram (not a rhombus) form a rectangle.

11. The bisectors of the angles of a rectangle (not a square) form a square.

12. If a diagonal bisects an angle of a parallelogram, the figure is a rhombus.

13. D is any point on the side AC of triangle ABC, and E, F, G, and H are the midpoints of AD, CD, BC, and AB respectively. Prove that $EFGH$ is a parallelogram.

14. If on opposite sides of line AB, the parallelograms $ABCD$ and $ABEF$ are drawn, then $CDFE$ is also a parallelogram.

15. If two opposite sides of a parallelogram are produced by the same length in opposite directions, another parallelogram can be formed by joining their ends to vertices of the original parallelogram.

16. If lines are drawn from the midpoints of two opposite sides of a quadrilateral (not a parallelogram) to the midpoints of the diagonals, these lines are parallel.

17. In the parallelogram $ABCD$, points E and F are taken on sides CB and AD respectively, so that $CE = AF$. Prove that $AECF$ and $BFDE$ are parallelograms.

18. Show how to place two congruent triangles so that a side of one shall coincide with the corresponding side of the other, and the figure formed by the two triangles shall be a parallelogram.

19. Prove that if the midpoints of the sides of a triangle are joined, four parallelograms are formed.

20. If two sides of a quadrilateral are parallel, and a pair of opposite angles are equal, the figure is a parallelogram.

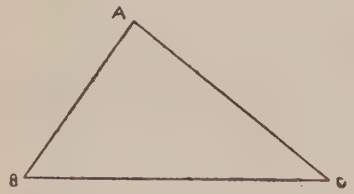
21. If two lines are equal and bisect each other, the lines joining their extremities form a rectangle.

22. If the diagonals of a trapezoid are equal, the trapezoid is isosceles.

23. If the line joining the midpoints of two sides of a triangle is produced by its own length, and the extremity of this line joined to the nearer end of the base, a parallelogram is formed whose area is equal to that of the triangle.

24. The lines joining the midpoints of the sides of a rectangle, taken in order, form a rhombus.

(I). *The sum of two sides of a triangle is greater than the third side, and their difference is less than the third side.*



Given $\triangle ABC$.

To prove (*a*) $AC+AB>BC$. (*b*) $AC-AB<BC$.

Proof

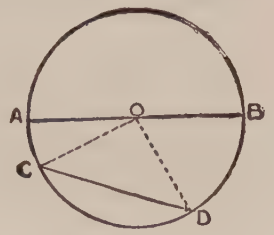
(*a*)

$AC+AB>BC$.

(*b*)

1. $AB+BC>AC$.
2. $AB=AB$.
3. $\therefore BC>AC-AB$.
4. *Or* $AC-AB<BC$.

(II). *A diameter of a circle is greater than any other chord.*



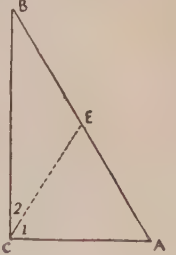
Given $\odot O$, diameter AB, and any other chord CD.

To prove $AB>CD$.

Proof

1. Draw radii OC and OD.
2. $OC+OD>CD$.
3. But $OA=OC$, and $OB=OD$.
4. $\therefore OA+OB>CD$.
5. $OA+OB=AB$.
6. $\therefore AB>CD$.

(I). *If an acute angle of a right triangle is 60°, the hypotenuse is double the shorter side.*



Given

rt. $\triangle ABC$, $\angle A=60°$, and $\angle C=90°$.

To prove

$AB=2AC$.

Proof

1. Draw CE, making $\angle 1=\angle A=60°$.
2. $\angle AEC=60°$.
3. $\angle B=30°$.
4. $\angle 2=30°$.
5. $\therefore \angle 2=\angle B$.
6. $BE=EC$.
7. $\triangle AEC$ is equilateral.
8. $AE=AC$, $EC=AC$.
9. $\therefore BE=AC$.
10. $AB=BE+AE$.
11. $\therefore AB=AC+AC=2AC$.

INEQUALITIES

EXERCISE 15

1. Two sides of a triangle are 3″ and 8″ respectively. Between what numerical limits does the length of the third side lie?

2. The shortest distance from a point to a circle is measured along a line which is drawn from the point through the center of the circle.

3. The sum of the diagonals of any quadrilateral is less than the perimeter, but greater than the semi-perimeter of the quadrilateral.

4. The sum of the four lines drawn to the vertices of any quadrilateral from any point within or without the quadrilateral, except the intersection of the diagonals, is greater than the sum of the diagonals.

5. If in triangle ABC, $AC = BC$, and a point D in the prolongation of AC is connected with B, then $DA > DB$.

6. The longest line from a point within a circle to the circle is the longer segment of the diameter passing through the point.

7. The shortest line from a point within a circle to the circle is the shorter segment of the diameter passing through the point.

8. Any point not on the perpendicular bisector of a line is unequally distant from the ends of the line.

9. Two adjacent sides of a parallelogram are 11 in. and 15 in. respectively. Between what limiting values must the lengths of the diagonals lie?

10. The perimeter of any polygon inscribed in a circle is less than the circumference of the circle.

11. Each side of a triangle is less than the semi-perimeter of the triangle.

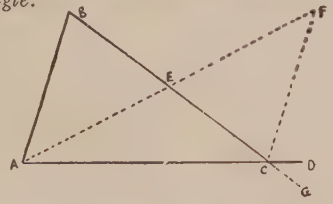
12. If the opposite ends of any two non-intersecting line-segments are joined, the sum of the joining lines is greater than the sum of the other two lines.

13. Any point within an angle, not on the bisector of the angle, is unequally distant from the sides of the angle.

14. P, Q, R, and S, are any points in the sides AB, BC, CD, and DA, respectively, of quadrilateral $ABCD$. Prove that the perimeter of $PQRS$ is less than that of $ABCD$.

15. Within a triangle ABC, a point P is taken so that $CP = CB$. Prove that $AB > AP$.

(I). *An exterior angle of a triangle is greater than either remote interior angle.*



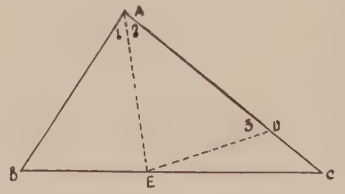
Given $\triangle ABC$ and ext. $\angle DCB$.

To prove $\angle DCB > \angle A$ or $\angle B$.

Proof

1. Let E be the midpoint of BC. Draw AE, and produce it its own length to F. Draw FC.
2. $AE = EF$, and $BE = EC$.
3. $\angle BEA = \angle CEF$.
4. $\triangle ABE \cong \triangle CEF$.
5. $\therefore \angle B = \angle ECF$.
6. But $\angle DCB > \angle ECF$.
7. $\therefore \angle DCB > \angle B$.
8. In the same way, by prolonging BC to G, and joining B to the midpoint of AC, we can prove that $\angle ACG > \angle A$.
9. But $\angle ACG = \angle DCB$.
10. $\therefore \angle DCB > \angle A$.

(I). *If two sides of a triangle are unequal, the angle opposite the greater side is greater than the angle opposite the smaller side.*



Given $\triangle ABC$ with $AC > AB$.

To prove $\angle B > \angle C$.

Proof

1. Draw AE bisecting $\angle BAC$. Let $AD = AB$, and draw ED.
2. $AD = AB$, and $\angle 1 = \angle 2$.
3. $AE = AE$.
4. $\triangle ABE \cong \triangle ADE$.
5. $\angle 3 = \angle B$.
6. But $\angle 3 > \angle C$.
7. $\therefore \angle B > \angle C$.

(I). *If two angles of a triangle are unequal, the side opposite the greater angle is greater than the side opposite the smaller angle.*

Given $\triangle ABC$ with $\angle B > \angle C$.

To prove $AC > AB$.

Proof

1. Either $AC = AB$, *or* $AC < AB$, *or* $AC > AB$.
2. If $AC = AB$, then $\angle B = \angle C$.
3. But this is impossible.
4. $\therefore AC \neq AB$.
5. If $AC < AB$, then $\angle B < \angle C$.
6. This is impossible.
7. $\therefore AC$ is not less than AB.
8. Hence, $AC > AB$.

30°—60° TRIANGLES

EXERCISE 16

1. In triangle ABC, $\angle B = 2\angle A$, and $\angle C$ is a right angle. If BA is 10 inches long, how long is BC?

2. In quadrilateral $ABCD$, AB and BC are each 8 inches long. $\angle A = 120°$, $\angle B = 60°$, and $\angle C = 150°$. Find the length of AD.

3. In trapezoid $ABCD$, $AB = BC = 28$ inches, and $\angle B = 60°$. AD is perpendicular to AB. Find the length of DC.

4. The triangle ABC has a right angle at C, and point D trisects AC, so that $AD = 2DC$. It is then found that $AD = BD$. Find the number of degrees in angle A.

5. ABC is an equilateral triangle. On the sides AB, BC, and CA lay off AD, BE, and CF, each equal to $\frac{1}{3}$ AB. Join the points D, E, and F. Prove that the triangle DEF is equilateral, and that its sides are respectively perpendicular to the sides of the given triangle.

6. The area of a triangle having an angle of 30° is one-fourth the product of the sides including this angle.

7. Two sides of a parallelogram are 10 inches and 18 inches respectively, and they make an angle of 60° with each other. Find the area of the parallelogram.

8. The vertex angle of an isosceles triangle is 120°. Prove that the altitude to the base is equal to one-half either arm.

9. If two tangents to a circle meet at an angle of 120°, the sum of the tangents is equal to the line joining the point of intersection of the tangents and the center of the circle.

10. In $\triangle ABC$, $\angle A = 60°$, $AB = 12$ in., and $\angle C$ is a right angle. If CD is the altitude to AB, find the length of AD.

11. Two sides of a parallelogram are 20 in. and 12 in. respectively, and the included angle is 30°. Find the area of the parallelogram.

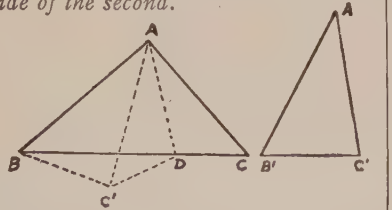
12. From a given point P outside a given line AB construct a line terminating in AB whose length shall be equal to twice its projection on AB.

13. In isosceles triangle ABC, $AB = BC$ and $\angle C = 30°$. Prove that the altitude to either arm of the triangle is equal to one-half AC.

14. $ABCD$ is a trapezoid. Lower base $AB = BC = 18$ in., $\angle A = 90°$ and $\angle B = 60°$. Find the length of upper base DC.

15. In isosceles triangle ABC, $\angle B = \angle C = 15°$, and $AB = 12$ in. Find the length of the altitude to an arm of the triangle.

(I). *If two triangles have two sides of one equal respectively to two sides of the other, but the included angle of the first greater than the included angle of the second, then the third side of the first is greater than the third side of the second.*



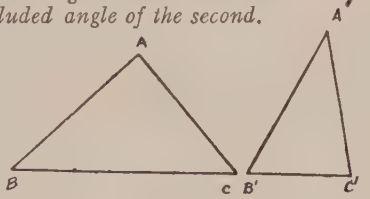
Given $\triangle ABC$ and $A'B'C'$ with $AB = A'B'$, $AC = A'C'$, and $\angle A > \angle A'$.

To prove $BC > B'C'$.

Proof

1. Place $\triangle A'B'C'$ on $\triangle ABC$ so that $A'B'$ coincides with its equal AB.
2. $A'C'$ will fall between AB and AC, as AC'.
3. Let AD bisect $\angle CAC'$. Draw DC'.
4. $AC = AC'$. **5.** $AD = AD$.
6. $\angle C'AD = \angle CAD$.
7. $\triangle C'AD \cong \triangle CAD$. **8.** $\therefore DC' = DC$.
9. But $BD + DC' > BC'$.
10. $\therefore BD + DC > BC'$.
11. $\therefore BC > BC'$. **12.** *i.e.*, $BC > B'C'$.

(I). *If two triangles have two sides of one equal respectively to two sides of the other, but the third side of the first greater than the third side of the second, then the included angle of the first is greater than the included angle of the second.*



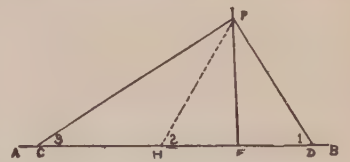
Given $\triangle ABC$ and $A'B'C'$ with $AB = A'B'$, $AC = A'C'$, and $BC > B'C'$.

To prove $\angle A > \angle A'$.

Proof

1. Either $\angle A = \angle A'$ or $\angle A < \angle A'$ or $\angle A > \angle A'$.
2. If $\angle A = \angle A'$, then $\triangle ABC \cong \triangle A'B'C'$.
3. BC would then be equal to $B'C'$.
4. This is impossible.
5. $\therefore \angle A \neq \angle A'$.
6. If $\angle A < \angle A'$, then $BC < B'C'$.
7. This is impossible.
8. $\therefore \angle A > \angle A'$.

(I). *If from a point in a perpendicular to a line, oblique lines are drawn cutting off unequal distances from the foot of the perpendicular, the one cutting off the greater distance is the greater.*



Given line $PF \perp AB$ at F, P any point in PF, and $FC > FD$.

To prove $PC > PD$.

Proof

1. On FC, take $FH = FD$, and draw PH.
2. $PH = PD$. **3.** $\angle 1 = \angle 2$.
4. $\angle 2 > \angle 3$. **5.** $\therefore \angle 1 > \angle 3$.
6. $\therefore PC > PD$.

Cor. *If from a point in a perpendicular to a line, unequal oblique lines are drawn, the greater cuts off the greater distance from the foot of the perpendicular.* (*Prove by the indirect method*).

INEQUALITIES—*Continued*

SUMMARY

1. The sum of two sides of a triangle is greater than the third side, and their difference is less than the third side.

2. An exterior angle of a triangle is greater than either remote interior angle.

3. In equal circles, if two chords are unequal, the greater chord is nearer the center; and conversely.

4. In equal circles, if two arcs are unequal, the greater arc has the greater chord; and conversely.

Note.—If two lines or angles in the same triangle are to be proved unequal, use **5** or **6** below.
If the lines or angles to be proved unequal are in different triangles, use **7** or **8** below.

5. If two sides of a triangle are unequal, the angle opposite the greater side is greater than the angle opposite the smaller side.

6. If two angles of a triangle are unequal, the side opposite the greater angle is greater than the side opposite the smaller angle.

7. If two triangles have two sides of one equal to two sides of the other, but the included angle of the first greater than the included angle of the second, then the third side of the first is greater than the third side of the second.

8. If two triangles have two sides of one equal to two sides of the other, but the third side of the first greater than the third side of the second, then the angle opposite the third side of the first is greater than the angle opposite the third side of the second.

EXERCISE 17

1. In quadrilateral $ABCD$, $AD = BC$ and $\angle C < \angle D$. Prove that diagonal $AC >$ diagonal BD.

2. In quadrilateral $ABCD$, $AB = CD$, and $\angle B > \angle C$. Prove $\angle D > \angle A$.

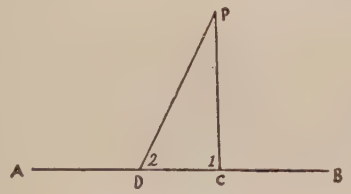
3. In $\triangle ABC$, $AC > BC$. On AC and BC respectively, equal segments AE and BD are laid off. Prove that $AD > BE$.

4. In $\triangle ABC$, BD is drawn bisecting $\angle B$ and meeting AC at D. Prove that $BA > AD$.

5. In the diagram of the preceding exercise, prove that $BC > DC$.

6. In quadrilateral $ABCD$, AB is the longest side and CD is the shortest side. Prove that $\angle D > \angle B$.

(I). *The perpendicular is the shortest line that can be drawn from a point to a given line.*



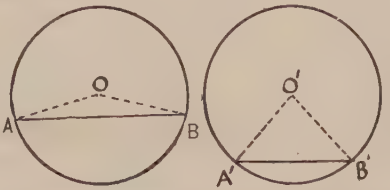
Given point P outside line AB, $PC \perp AB$, and PD any other line from P to AB.

To prove $PC < PD$.

Proof

1. $\angle 1$ is a rt. $\angle$.
2. $\therefore \angle 2 <$ a rt. $\angle$.
3. $\therefore \angle 1 > \angle 2$.
4. $\therefore PC < PD$.

(II). *In the same circle, or in equal circles, the greater of two unequal minor arcs has the greater chord; and conversely, the greater of two unequal chords has the greater minor arc.*



Given equal ⓢ O and O' and $\overset{\frown}{AB} > \overset{\frown}{A'B'}$.

To prove chord $AB > A'B'$.

Proof

1. Draw OA, OB, $O'A'$, and $O'B'$.
2. In $\triangle AOB$ and $A'O'B'$, $AO = A'O'$, and $BO = B'O'$.
3. $\angle O > \angle O'$. 4. $\therefore AB > A'B'$.

Converse

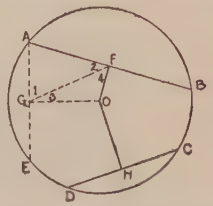
Given equal ⓢ O and O', and $AB > A'B'$.

To prove $\overset{\frown}{AB} > \overset{\frown}{A'B'}$.

Proof

1. Draw OA, OB, $O'A'$, and $O'B'$.
2. In $\triangle AOB$ and $A'O'B'$, $OA = O'A'$, and $OB = O'B'$. 3. $AB > A'B'$.
4. $\therefore \angle O > \angle O'$. 5. $\therefore \overset{\frown}{AB} > \overset{\frown}{A'B'}$.

(II). *In the same circle, or in equal circles, if two chords are unequal, the greater chord is nearer the center.*



Given $\odot O$ with $AB > CD$, $OF \perp AB$, and $OH \perp CD$.

To prove $OF < OH$.

Proof

1. Draw $AE = DC$, $OG \perp AE$, and FG.
2. $AB > CD$. 3. $\therefore AB > AE$.
4. $AF = \frac{1}{2}AB$; $AG = \frac{1}{2}AE$.
5. $AF > AG$. 6. $\therefore \angle 1 > \angle 2$.
7. $\angle AFO = \angle AGO$.
8. $\therefore \angle 3 < \angle 4$. 9. $OF < OG$.
10. But $OG = OH$. 11. $\therefore OF < OH$.

Cor. *In the same circle, or in equal circles, if two chords are unequally distant from the center, the chord nearer the center is the greater.* (*Prove by the indirect method.*)

INEQUALITIES—*Concluded*

EXERCISE 18

1. In equilateral triangle ABC, side AC is produced to D, and DB is drawn. Prove that $\angle A > \angle D$.

2. A line drawn from the vertex of an isosceles triangle to any point in the base is less than either arm of the triangle.

3. The diagonals of a rhombus (not a square) are unequal.

4. E is any point in side AB of triangle ABC, and EC is drawn. D is any point in EC, and BD is drawn. Prove that $\angle BDC > \angle BAC$.

5. Is it possible to draw a triangle whose sides are (a) 5 in., 6 in., and 12 in.? (b) 5 in., 6 in., and 11 in.? (c) 5 in., 6 in., and 10 in.?

6. In $\triangle ABC$, AB is greater than AC. Which side is greatest, and which is smallest, (a) if $\angle B = 60°$? (b) if $\angle A = 60°$?

7. A median of a triangle is less than half the sum of the two adjacent sides. (*Hint.*—Prolong the median its own length.)

8. In quadrilateral $ABCD$, $AB > AD$, and $\angle B = \angle D$. Prove that $CD > BC$.

9. Equal segments, AD and AE, are laid off on the sides of $\angle BAC$. P is any point not in the bisector of $\angle BAC$. Prove that PD and PE are unequal.

10. If a trapezoid is not isosceles, its base angles are unequal.

11. In an isosceles trapezoid, an angle adjacent to the smaller base is greater than an angle adjacent to the longer base.

12. In $\triangle ABC$, $AC > AB$. Let P be any point in median AD. Draw PB and PC. Prove $PC > PB$.

13. In $\triangle ABC$, altitude CD is drawn, and any point P on the altitude is connected with A and B. Prove $CA + CB > PA + PB$.

14. Given $\triangle ABC$ with $AC > BC$. The bisectors of angles A and B meet in O. Prove $AO > BO$.

15. B is 50 miles west of A, C is 40 miles north of B, and D is 40 miles southeast of B. Show that C is at a greater distance from A than D.

16. Two secants AB and AC are drawn to a circle, and AB passes through the center. Prove that $\angle ACB > \angle ABC$.

17. If an arm AB of isosceles triangle ABC is equal to an arm $A'B'$ of isosceles triangle $A'B'C'$, and base angle $B >$ base angle B', prove that base $B'C' >$ base BC.

18. D is any point in side AB of $\triangle ABC$. Prove that $AB + BC > AD + DC$. (*Hint.*—$DB + BC > DC$.)

19. In parallelogram $ABCD$, E and F are the midpoints of sides DC and BC, respectively. If $\angle C$ is increased, BC and DC remaining fixed in length, show that EF also increases. Between what limits will the length of EF always lie?

EXERCISE 19

1. If the diagonals of an inscribed quadrilateral are unequal, the opposite sides are unequal.

2. A, B, C, and D are points taken in succession on a semicircle, and $\overset{\frown}{AC} > \overset{\frown}{BD}$. Prove that chord $AB >$ chord CD.

3. If ABC is any triangle inscribed in a circle, and $\angle A > \angle B$, prove that $\overset{\frown}{BC} > \overset{\frown}{AC}$.

4. State and prove the converse of Ex. 2, above.

5. In $\triangle ABC$, AD is the altitude on BC. P is any point between B and D. Prove that AP is greater than AD, and less than AB.

6. The side of a regular octagon inscribed in a circle is less than the radius of the circle.

7. A side of an equilateral triangle inscribed in a circle is nearer the center than a side of an inscribed square.

8. The sum of the altitudes of a triangle is less than its perimeter.

9. If through a point in a circle, two chords are drawn making unequal angles with the radius drawn to the point, the chords are unequal.

10. Prove the correctness of the following method of drawing the shortest possible chord through a given point within a circle: Draw a radius through the given point, and at the given point construct a perpendicular to the radius.

11. Side BC of square $ABCD$ is extended to P, and PA is drawn. Prove that $PA >$ diagonal BD.

12. *Prove:* A diagonal of a parallelogram (not a rhombus) divides two opposite angles into unequal parts, the greater part being that angle which the diagonal forms with the shorter side.

13. In $\triangle ABC$, $AB = AC$, and D is any point in AB. Prove that $DC > DB$.

14. In $\triangle ABC$, AD is the median to BC, and $\angle ADB$ is an acute angle. Prove: (a) $AC > AB$; (b) $\angle B > \angle C$.

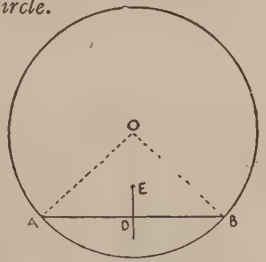
15. If the angle between two adjacent sides of a parallelogram is increased, show that the length of one diagonal increases, and the length of the other diagonal decreases.

16. In quadrilateral $ABCD$, AD is the longest side and BC is the shortest side. Prove that $\angle C > \angle A$.

17. Quadrilateral $ABCD$ has unequal sides, and E is the midpoint of diagonal AC. Prove that BE makes unequal angles with AC.

18. Side BC of equilateral triangle ABC is extended to point D and DA is drawn. Prove that $DA >$ CB.

(II). *The perpendicular bisector of a chord passes through the center of the circle.*



Given $\odot O$, chord AB, and DE, the $\perp$ bisector of AB.

To prove DE passes through O.

Proof

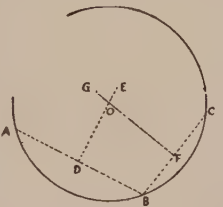
1. Draw OA and OB. **2.** $OA = OB$.
3. $\therefore DE$ passes through O.

Cor. 1. *A circle cannot be drawn through three points which lie in the same straight line.*

Cor. 2. *A straight line cannot intersect a circle in more than two points.*

Cor. 3. *Two circles cannot meet in more than two points.*

(II). *Through three points not lying in the same straight line, one circle and only one can be drawn.*



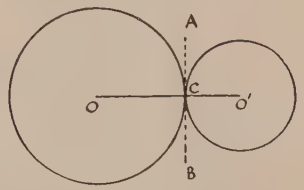
Given A, B, and C not in the same st. line.

To prove (*a*) a circle can be drawn through A, B, and C; (*b*) only one such circle can be drawn.

Proof

1. Draw AB and BC.
2. Draw DE, the $\perp$ bisector of AB, and FG, the $\perp$ bisector of BC.
3. Let DE and FG intersect in O.
4. $OA = OB$ and $OB = OC$.
5. $\therefore OA = OB = OC$.
6. *i.e.*, A, B, and C will lie on a circle whose center is O, and radius OA.
7. If another circle could be drawn through A, B, and C, its center would lie in DE and in FG.
8. $\therefore O$ is the only possible center.
9. OA is the only possible radius.
10. $\therefore$ only one circle can be drawn through A, B, and C.

(II). *If two circles are tangent to each other, their line of centers passes through the point of contact.*



Given ⊙ O and O', tangent at C, and their line of centers OO'.

To prove OO' passes through C.

Proof

1. Draw the common tangent AB at C.

2. The perpendicular to AB at C passes through O and O'.

3. $\therefore OO'$, the line of centers, will coincide with this perpendicular and pass through C.

LINES PASSING THROUGH SPECIAL POINTS

SUMMARY

1. The perpendicular bisector of a line contains all points equidistant from the ends of the line.

2. The bisector of an angle contains all points equidistant from the sides of the angle.

3. The perpendicular bisector of a chord passes through the center of the circle.

4. The line of centers of two tangent circles passes through the point of contact.

5. A point lies on a circle if its distance from the center is equal to the length of a radius.

EXERCISE 20

1. The perpendicular bisector of the base of an isosceles triangle passes through the vertex.

2. The bisector of the angle between two tangents passes through the center of the circle.

3. The bisectors of the angles of a circumscribed quadrilateral are concurrent.

4. AB and CD are unequal parallel chords. Prove that AD and BC will intersect on the diameter perpendicular to the chords, or on this diameter produced. (*Note.*—There are two possible solutions.)

5. AB is the common chord of two intersecting circles, and AC and AD are diameters. Prove that CD passes through B.

6. If a median of a triangle is half the side to which it is drawn, the triangle is a right triangle. (*Hint.*—Draw a circle having the midpoint of the side as center, and the median as radius.)

7. In quadrilateral $ABCD$, $AB = BC$ and $AD = DC$. If E is the midpoint of AC, prove that DE bisects $\angle B$.

8. *Prove:* The line joining the midpoints of two parallel chords passes through the center of the circle.

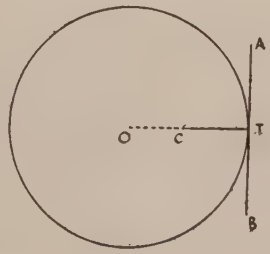
9. The line joining the midpoints of an arc of a circle and its chord is a diameter.

10. If an equilateral polygon is inscribed in a circle, prove that a second circle can be inscribed in the polygon.

11. The perpendicular bisector of either base of an isosceles trapezoid passes through the point of intersection of the non-parallel sides, produced.

12. The circle inscribed in an equilateral triangle and the circle circumscribed about the same triangle are concentric; i.e., have a common center.

(II). *The perpendicular to a tangent at the point of contact passes through the center of the circle.*



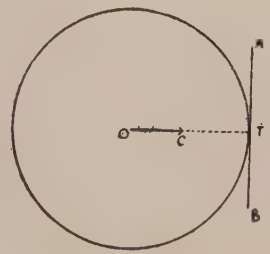
Given AB tangent to circle O at T, and $TC \perp AB$.

To prove TC passes through O.

Proof

1. Connect T and O.
2. TO is $\perp AB$.
3. But TC is $\perp AB$.
4. $\therefore TO$ and TC coincide, and TC passes through O.

(II). *The perpendicular from the center of a circle to a tangent meets it at the point of contact.*



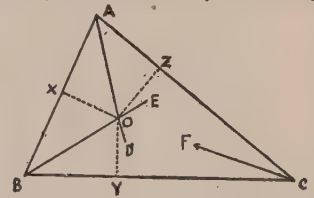
Given AB tangent to $\odot O$ at T, and $OC \perp AB$.

To prove OC passes through T.

Proof

1. Connect O and T.
2. $OT \perp AB$.
3. But $OC \perp AB$.
4. $\therefore OC$ and OT coincide, and OC passes through T.

(I). *The bisectors of the angles of a triangle meet in a point which is equidistant from the sides of the triangle.*



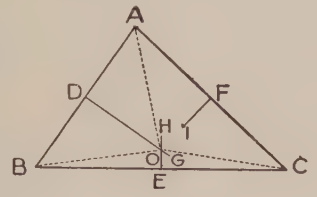
Given $\triangle ABC$, and AD, BE, and CF bisecting $\measuredangle A$, B, and C, respectively.

To prove (*a*) AD, BE, and CF meet in a point, such as O; (*b*) O is equidistant from AB, BC, and AC.

Proof

1. If AD and BE were $\parallel$, $\measuredangle BAD$ and ABE would be supplementary.
2. But $\measuredangle BAD$ and ABE are not supplementary.
3. $\therefore AD$ and BE are not $\parallel$. Let their meeting point be O. Draw OX, OY, and $OZ \perp$ respectively to AB, BC, and AC.
4. $OX = OY$; $OX = OZ$.
5. $\therefore OZ = OY$.
6. CF passes through O.
7. $\therefore AD$, BE, and CF meet in O, and O is equidistant from AB, BC, and AC.

(I). *The perpendicular bisectors of the sides of a triangle meet in a point which is equidistant from the vertices of the triangle.*



Given $\triangle ABC$ with DG, FI, and $HE \perp$ bisectors of AB, AC, and BC respectively.

To prove (*a*) DG, FI, and HE meet in a point, such as O; (*b*) O is equidistant from A, B, and C.

Proof

1. DG and HE will meet.

2. Let O be the meeting point of DG and HE. Draw OA, OB, and OC.

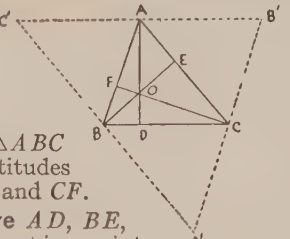
3. $OA = OB$ and $OB = OC$.

4. $\therefore OA = OC$.

5. FI passes through O.

6. $\therefore DG$, FI, and HE meet in O, and O is equidistant from A, B, and C.

(I). *The three altitudes of a triangle (produced if necessary) meet in a point.*



Given $\triangle ABC$ with altitudes AD, BE, and CF.

To prove AD, BE, and CF meet in a point.

Proof

1. Through the vertices of $\triangle ABC$, draw $B'C' \| BC$, $A'B' \| AB$, and $A'C' \| AC$.

2. $ABCB'$ and $ACBC'$ are $\square$.

3. $AB' = BC$, and $AC' = BC$.

4. $\therefore AC' = AB'$. **5.** $AD \perp BC$.

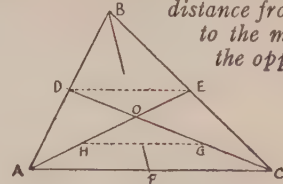
6. $\therefore AD \perp B'C'$.

7. $\therefore AD$ is the $\perp$ bisector of $B'C'$.

8. In the same way, we can prove that CF is the $\perp$ bisector of $A'B'$, and BE the $\perp$ bisector of $A'C'$.

9. $\therefore AD$, BE, and CF meet in a point.

(I). *The medians of a triangle meet in a point which is two-thirds of the distance from a vertex to the midpoint of the opposite side.*



Given $\triangle ABC$ with medians AE, CD, and BF.

To prove AE, CD, and BF meet in a point such as O, and $AO = \frac{2}{3}AE$, $CO = \frac{2}{3}CD$, and $BO = \frac{2}{3}BF$.

Proof

1. If AE and CD were parallel, $\angle EAC + \angle DCA$ would be equal to 180°.

2. But $\angle EAC + \angle DCA$ is not equal to 180°.

3. $\therefore AE$ and CD are not parallel.

4. Let AE and CD meet in O.

5. Let H and G be the midpoints of AO and CO respectively. Draw DE and HG.

6. $DE \| AC$ and $= \frac{1}{2}AC$. $HG \| AC$ and $= \frac{1}{2}AC$.

7. $DE = HG$. **8.** $DE \| HG$. **9.** $\angle DEO = \angle OHG$ and $\angle EDO = \angle OGH$.

10. $\triangle DOE \cong \triangle OHG$. **11.** $\therefore HO = OE$ and $DO = OG$.

12. But $HO = AH$ and $OG = GC$. **13.** $AH = HO = OE$, and $GC = OG = OD$.

14. $\therefore AO = \frac{2}{3}AE$ and $CO = \frac{2}{3}CD$.

15. In the same way, we can prove that BF passes through O, the trisection point of CD and AE, and that $BO = \frac{2}{3}BF$.

16. $\therefore AE$, CD, and BF meet in point O, and $AO = \frac{2}{3}AE$; $CO = \frac{2}{3}CD$, and $BO = \frac{2}{3}BF$.

LINES PASSING THROUGH SPECIAL POINTS

(*Concluded*)

EXERCISE 21

1. Find, by construction, the center of a given arc.

2. A circle can be circumscribed about any regular polygon. (*Hint.*—Draw a circle through three vertices of the polygon and prove that it will pass also through the remaining vertices.)

3. The bisectors of two exterior angles of a triangle and the bisector of the remote interior angle meet in a point.

4. Given two lines that would meet if sufficiently produced, draw the bisector of their angle without producing the lines.

5. The line joining the points of tangency of two parallel tangents passes through the center of the circle.

6. If an isosceles triangle is inscribed in a circle, prove that the bisector of the vertex angle passes through the center of the circle.

7. Three equal circles are tangent to each other. Prove that the three common tangents through their points of contact meet in a point which is equidistant from the centers of the circles. (*Hint.*—Draw the three center lines.)

8. The bisectors of all angles inscribed in a given segment are concurrent.

9. The bisectors of the angles of a circumscribed polygon are concurrent.

10. The centers of all circles that can be drawn tangent to a given line AB at a given point P in the line are co-linear; *i.e.*, they all lie on the same straight line.

11. In quadrilateral $ABCD$, the perpendicular bisectors of AB and AD meet in point O, and the perpendicular bisectors of BC and CD meet in point O'. Prove that OO' is the perpendicular bisector of BD.

12. In $\triangle ABC$, median AD is 8 in. long. Find the length of the segment of AD from A to the point of intersection of the medians.

13. In $\triangle ABC$, medians BE, AD, and CF meet in point O. G is the midpoint of BO, and FG is drawn. Prove that $FG = OD$.

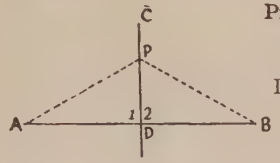
14. *Prove:* The line joining the center of a circle to the midpoint of a chord is perpendicular to the chord.

15. AB and CD are two unequal parallel chords in a circle. Prove that the line joining the middle points of these chords passes through the point of intersection of AD and BC, and also that of AC and BD

(I). *The perpendicular-bisector of a line is the locus of points equidistant from the ends of the line.*

Given line AB and its perpendicular-bisector CD.

To prove CD is the locus of points equidistant from A and B.



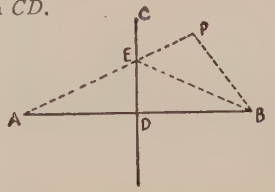
Proof

Part I

1. Let P be any point in CD. Draw PA and PB.
2. $PD = PD$.
3. $AD = DB$.
4. $\angle 1 = \angle 2$.
5. $\triangle PDA \cong \triangle PDB$.
6. $PA = PB$.
7. $\therefore$ Every point in CD is equidistant from A and B.

Part II

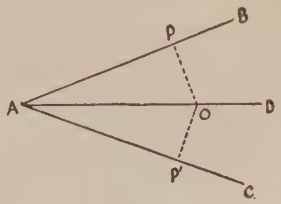
1. Let P be any point not in CD. Draw PA, PB and BE.
2. $PE + EB > PB$.
3. But $EB = EA$.
4. $\therefore PE + EA > PB$.
5. $PE + EA = PA$.
6. $\therefore PA > PB$.
7. *i.e.*, every point not in CD is unequally distant from A and B.
8. $\therefore$ CD is the locus of points equidistant from A and B.



(I). *The bisector of an angle is the locus of points within the angle equally distant from its sides.*

Given $\angle BAC$ and AD its bisector.

To prove AD is the locus of points within the angle equally distant from its sides.



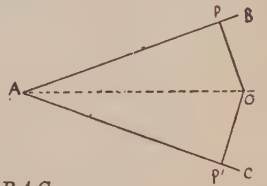
Proof

Part I

1. Let O be any point in AD. Draw $OP \perp AB$ and $OP' \perp AC$.
2. $OA = OA$.
3. $\angle OPA = \angle OP'A = 90°$.
4. $\angle OAP = \angle OAP'$.
5. $\triangle OAP \cong \triangle OAP'$.
6. $\therefore OP = OP'$.
7. $\therefore$ Every point in AD is equidistant from AB and AC

Part II

1. Let O be any point within the angle and equidistant from AB and AC. Draw OA.
2. Draw $OP \perp AB$, and $OP' \perp AC$.
3. $OP' = OP$. **4.** $OA = OA$.
5. Rt. $\triangle OAP \cong$ rt. $\triangle OAP'$.
6. $\angle OAP = \angle OAP'$.
7. $\therefore OA$ is the bisector of $\angle BAC$.
8. $\therefore$ Every point within the angle and equidistant from AB and AC lies in the bisector of the angle.
9. $\therefore$ The bisector of the angle is the locus of points within the angle equidistant from its sides.



LOCI

SUMMARY

To prove that a line is the locus of points which have some special property, prove 1 and either 2 or 3:

1. Every point on the line has the special property mentioned in the proposition.

2. Every point outside the line does not have the special property. (Opposite of 1.)

3. Every point that has the special property is on the line. (Converse of 1.)

EXERCISE 22

1. The locus of points equidistant from the ends of a given line is the perpendicular bisector of the line.

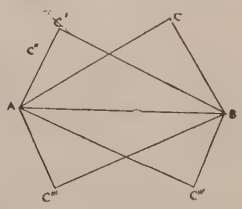
2. The locus of points equally distant from two intersecting lines consists of the four bisectors of the included angles.

3. The locus of points at a given distance from a given point is a circle having the given point as center and the given distance as radius.

4. The locus of points at a given distance from a given line of indefinite length consists of two lines parallel to the given line and at the given distance from it.

5. The locus of points equidistant from two parallel lines is the perpendicular bisector of a line ending in the two parallels and perpendicular to them.

Finding Loci.—To discover what line or lines constitute the locus of a point that has some special property, the student should locate by construction several of the points that have this special property. A line or lines drawn so as to include all such points will be the required locus. In elementary geometry, the locus will always be a straight line or a circle. If a circle, its center and radius must be found.

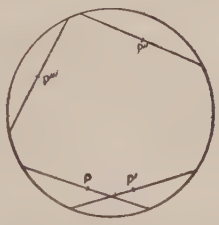


Example.—On a line of given length as hypotenuse, a right triangle is constructed. Find the locus of the vertex of the right angle.

Solution.—Let AB be the given hypotenuse. Then, C, C', C'', etc., are possible positions of the vertex of the right angle. A line drawn so as to include all the points C, C', C'', etc., is the required locus. In this case, it is a circle whose diameter is AB.

EXERCISE 23

1. Find the locus of the midpoints of all chords of given length that can be drawn in a given circle.



2. A right triangle has the vertex and the sides of its right angle fixed in position and the length of its hypotenuse constant. Find the locus of the midpoint of the hypotenuse.

3. Given a line segment whose length is 3 inches. Construct the locus of points one inch from the nearest point on the given segment.

4. State, without giving proof, the locus of each of the following:

(a) The centers of circles tangent to two given parallel lines.
(b) The centers of circles tangent to two intersecting lines.
(c) The centers of circles passing through two given points.
(d) The centers of circles tangent to a given circle at a given point in the circle.
(e) The centers of circles tangent to a given line at a given point in the line.
(f) The centers of circles tangent to a given line and having a given radius.
(g) The centers of circles having a given radius and tangent to a given circle (1) internally; (2) externally.
(h) The centers of circles tangent to each of two given concentric circles.

5. A circle of given radius rolls around the perimeter of a given equilateral triangle. Construct the locus of its center.

6. A right triangle ABC has a fixed hypotenuse AB, 2 inches long. The vertex of the right angle, C, is allowed to take all possible positions. Construct accurately, and describe fully, the locus of the midpoint, P, of the leg AC.

7. A circle whose radius is 5 inches contains a moving chord 8 inches long, which is divided into four equal parts by the points P, Q, and R. Find the loci of points P, Q, and R.

8. Two lines AB and AC, each 2 inches long, form right angle BAC. Construct the locus of the points whose least distance from the figure is 1 inch.

9. Find the locus of all points from which tangents of a given length can be drawn to a given circle.

10. Within a rectangle which is not a square, circles are drawn tangent to two sides of the rectangle. Find the locus of the centers of these circles.

11. P is any point on sides BC and CD of square $ABCD$. AP is drawn and produced its own length to Q. Find the locus of Q.

12. Find the locus of the midpoints of chords drawn through a fixed point within a circle.

13. A rod whose length is a is free to move within a semicircular area whose radius is a. Describe accurately the boundary of the region within which the middle point of the rod will always be found.

Note.—To find by construction a point or points that fulfill two given conditions, construct two separate loci, one satisfying each condition. Their points of intersection are the required points.

14. Find a point that is equidistant from two given points and also equidistant from two given parallel lines. Under what condition is this problem impossible?

15. Find all points that are equidistant from two given intersecting lines and also at a given distance from a given point. In this problem, there may be one, two, three, four, or no points possible. Explain.

16. A certain tree is 6 feet from a straight fence. If a pot of gold were buried 4 feet from the fence and 12 feet from the tree, show by construction in how many places it might be necessary to dig in order to find it.

17. Construct a circle with a given radius which shall pass through a given point and be tangent to a given straight line.

18. Find by construction all points at a given distance from a given circle and at the same distance from a given point.

19. Construct a circle tangent to two given lines and passing through a given point between the given lines. (*Two cases.*)

20. Given a circle and a line outside the circle. Construct a circle tangent to the given circle, having for its radius a given distance, and its center in the given line.

21. Given a straight line and two points on the same side of that line and at unequal distances from it. Construct a circle passing through the two points and having its center in the given line.

22. Construct a circle of given radius tangent to two given circles. Is more than one construction possible?

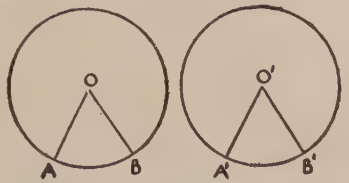
23. Find by construction a point which is equidistant from two given intersecting lines, and at the same time equidistant from two given points. Is it always possible to find such a point?

24. Two vertices, B and C, of a triangle have a fixed position. Find the locus of the third vertex A, if M_a is equal to a given line.

25. A line is drawn parallel to one side of a triangle, and having its ends in the other two sides. Find the locus of its middle point.

26. Find the locus of the vertex of a triangle which has a fixed base and a fixed vertex angle.

(II). *In the same circle, or in equal circles, equal central angles have equal arcs; and conversely, equal arcs have equal central angles.*



Given equal Ⓢ O and O' with equal central $\measuredangle$ O and O'.

To prove $\overset{\frown}{AB} = \overset{\frown}{A'B'}$.

Proof

1. Place $\odot O$ on $\odot O'$ so that OA coincides with its equal $O'A'$.
2. OB takes the direction of $O'B'$.
3. B coincides with B'.
4. $\overset{\frown}{AB}$ coincides with $\overset{\frown}{A'B'}$.
5. $\therefore \overset{\frown}{AB} = \overset{\frown}{A'B'}$.

Converse

Given equal Ⓢ O and O' with $\overset{\frown}{AB} = \overset{\frown}{A'B'}$. **To prove** $\angle O = \angle O'$.

Proof.—1. Place $\odot O$ on $\odot O'$ so that OA coincides with its equal $O'A'$.

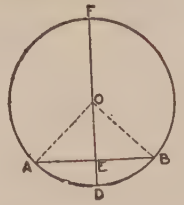
2. $\overset{\frown}{AB}$ falls along $\overset{\frown}{A'B'}$.
3. B falls on B'.
4. OB coincides with $O'B'$.
5. $\therefore \angle O = \angle O'$.

Cor. *In the same circle, or in equal circles, if two central angles are unequal, the greater angle has the greater arc; and conversely.*

(II). *The diameter perpendicular to a chord bisects the chord and the arcs of the chord.*

Given $\odot O$ and diameter $FD \perp AB$.

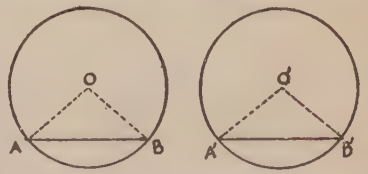
To prove $AE = EB$, $\overset{\frown}{AD} = \overset{\frown}{DB}$, and $\overset{\frown}{AF} = \overset{\frown}{BF}$.



Proof

1. Draw radii OA and OB.
2. $OA = OB$.
3. $OE = OE$.
4. $\measuredangle$ OEA and OEB are rt. $\measuredangle$.
5. $\triangle OEA \cong \triangle OEB$.
6. $\therefore AE = EB$, and $\angle AOE = \angle BOE$.
7. $\therefore \overset{\frown}{AD} = \overset{\frown}{DB}$.
8. $\overset{\frown}{FAD} = \overset{\frown}{FBD}$.
9. $\therefore \overset{\frown}{AF} = \overset{\frown}{BF}$.

(II). *In the same circle, or in equal circles, equal arcs have equal chords; and conversely, equal chords have equal arcs.*



Given equal Ⓢ O and O' with $\overset{\frown}{AB} = \overset{\frown}{A'B'}$.

To prove $AB = A'B'$.

Proof

1. Draw OA, OB, $O'A'$, and $O'B'$.
2. $\angle O = \angle O'$.
3. $AO = A'O'$, and $BO = B'O'$.
4. $\triangle AOB \cong \triangle A'O'B'$.
5. $\therefore AB = A'B'$.

Converse

Given equal Ⓢ O and O' and $AB = A'B'$.

To prove $\overset{\frown}{AB} = \overset{\frown}{A'B'}$.

Proof

1. Draw OA, OB, $O'A'$, and $O'B'$.
2. $AB = A'B'$.
3. $OA = O'A'$, and $OB = O'B'$.
4. $\triangle BOA \cong \triangle B'O'A'$.
5. $\angle O = \angle O'$.
6. $\therefore \overset{\frown}{AB} = \overset{\frown}{A'B'}$.

EQUAL ARCS

SUMMARY

To prove two arcs equal, in the same circle or in equal circles, prove one of the following:

1. Their central angles are equal.
2. Their chords are equal.
3. The angles measured by the arcs are equal.
4. The arcs are intercepted by parallel lines.

EXERCISE 24

1. If the lines which join a point in a circle to the midpoints of two radii are equal, the point bisects the arc intercepted by the radii.

2. Two equal circles intersect at A and B. Through A and B, parallel lines CAD and EBF are drawn, meeting the circles at C, D, and E, F, respectively. Prove $\overset{\frown}{CE} = \overset{\frown}{DF}$.

3. From a point A outside a circle equal secants ABD and ACE are drawn, cutting the circle at B, D, and C, E, respectively. Prove that the line bisecting $\angle DAE$ bisects also arcs DE and BC.

4. Prove the preceding theorem when A is inside the circle. Is the theorem true when A is on the circle?

5. If a secant intersects two concentric circles, its segments, intercepted by the two circles, are equal.

6. The diameter of a given circle is the radius of a second circle. Prove that all chords of the larger circle, drawn from the point of tangency of the two circles, are bisected by the smaller circle.

7. If $\triangle ABC$ is inscribed in a circle, and $\angle A = \angle B$, prove that $\overset{\frown}{AC} = \overset{\frown}{BC}$.

8. If a diameter and another chord are drawn from a point in a circle, the arc intercepted by the angle between them is bisected by a diameter drawn parallel to the chord.

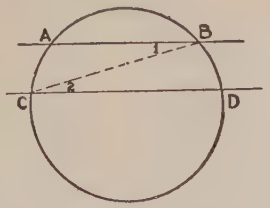
9. If the perpendiculars drawn from a point in a circle upon two radii are equal, the point bisects the arc intercepted by the two radii.

10. $ABCD$ is a trapezoid inscribed in a circle. Prove $\overset{\frown}{DAB} = \overset{\frown}{ABC}$.

11. If through a point, equidistant from two points in a circle, a radius is drawn, the arc between the two points is bisected.

12. Show how to divide a circle into eight equal parts.

(II). *Parallel secants intercept equal arcs on a circle.*



Given two parallel secants, AB and CD.

To prove $\overset{\frown}{AC} = \overset{\frown}{BD}$.

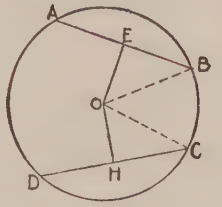
Proof

1. Draw BC.
2. $\angle 1 = \angle 2$.
3. $\angle 1$ is meas. by $\frac{1}{2}\overset{\frown}{AC}$, and $\angle 2$ is meas. by $\frac{1}{2}\overset{\frown}{BD}$.
4. $\therefore \frac{1}{2}\overset{\frown}{AC} = \frac{1}{2}\overset{\frown}{BD}$.
5. $\therefore \overset{\frown}{AC} = \overset{\frown}{BD}$.

Cor. 1. *If a secant is parallel to a tangent, they intercept equal arcs on the circle.*

Cor. 2. *Two parallel tangents intercept equal arcs on the circle.*

(II). *In the same circle, or in equal circles, equal chords are equally distant from the center.*



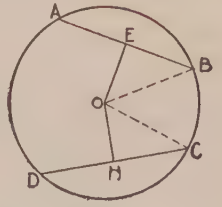
Given $\odot O$, with $AB = CD$, $OE \perp AB$ and $OH \perp CD$.

To prove $OE = OH$.

Proof

1. Draw OB and OC.
2. $AB = CD$, $OE \perp AB$, and $OH \perp CD$.
3. $EB = \frac{1}{2}AB$, and $CH = \frac{1}{2}CD$.
4. $\therefore EB = CH$.
5. $OB = OC$.
6. $\triangle BEO$ and OHC are rt. $\triangle$.
7. $\therefore \triangle BEO \cong \triangle OHC$.
8. $\therefore OE = OH$.

(II). *In the same circle, or in equal circles, chords equally distant from the center are equal.*



Given $\odot O$, with $OE \perp AB$, $OH \perp CD$, and $OE = OH$.

To prove $AB = CD$.

Proof

1. Draw OB and OC.
2. $OE = OH$.
3. $OB = OC$.
4. $\triangle OBE$ and OHC are rt. $\triangle$.
5. $\therefore \triangle OBE \cong \triangle OHC$.
6. $\therefore EB = HC$.
7. But $EB = \frac{1}{2}AB$ and $CH = \frac{1}{2}CD$.
8. $\therefore AB = CD$.

EQUAL CHORDS

SUMMARY

In the same circle or in equal circles, chords are equal if—

1. They have equal arcs.
2. They are equally distant from the center.

EXERCISE 25

1. AB is a diameter of a circle, and chords AC and BD are parallel. Prove $AC = BD$.

2. From a point A outside a circle, equal secants ABC and ADE are drawn. Prove that the external segments of the secants are equal.

3. If a straight line bisects a chord and one of its subtended arcs, it is perpendicular to the chord.

4. The diagonals of an equilateral pentagon inscribed in a circle are equal.

5. A straight line is drawn cutting two equal circles and parallel to their line of centers. Prove that the chords intercepted in the two circles are equal.

6. Base AB of isosceles triangle ABC is a chord of a circle which cuts the equal sides at F and G respectively. Prove that $AF = BG$. (*Two cases.*)

7. Two intersecting chords are equal if they make equal angles with the radius drawn through their intersection point.

8. If two chords of a circle, drawn from the same point on the circle, make equal angles with the tangent at that point, the chords are equal.

9. Two chords perpendicular to a third chord at its extremities are equal.

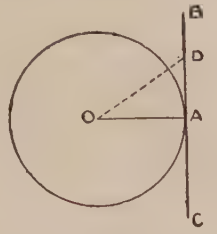
10. If ABE and DCE are two secants of a circle which make equal angles with the line connecting E with the center, then chord AB = chord DC.

11. An inscribed trapezoid is isosceles.

12. If the bisector of an inscribed angle is produced until it meets the circle, and through this point of intersection a chord is drawn parallel to one side of the angle, it is equal to the other side.

13. In a circle whose center is O, two perpendiculars OD and OE are drawn upon the chords AB and CF respectively. $\angle DEO = \angle EDO$. Prove that chord $AB =$ chord CF.

(II). *A straight line perpendicular to a radius at its outer extremity is tangent to the circle.*



Given $\odot O$, with $BC \perp OA$ at A.

To prove BC is a tangent.

Proof

1. Let D be any point in BC except A. Draw OD.

2. $OD > OA$.

3. $\therefore$ D lies without the $\odot$.

4. *i.e.*, every point in BC, except A, lies outside the $\odot$.

5. $\therefore$ BC is a tangent.

To prove a line tangent to a circle, prove that it is perpendicular to a radius at the outer end of the radius.

Note.—In exercises involving tangent circles, it is often useful to draw a common tangent to the circles.

EXERCISE 26

1. AOD is a diameter of a circle whose center is O. B is any point on the circle. At B a tangent is drawn. OP is drawn through the center, parallel to AB, meeting the tangent at P. Prove that PD is tangent to the circle. (*Hint.*—Draw BO).

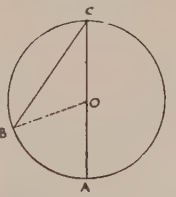
2. A common tangent to two unequal circles intersects the line of centers at P. From P a second tangent is drawn to one of the circles. Prove that it is also tangent to the other circle.

3. Two concentric circles are drawn, and chord AB of the outer circle is tangent to the inner circle. Prove that any other chord of the outer circle which is equal in length to AB is tangent to the inner circle.

4. In triangle ABC, the bisectors of angles B and C meet in point O. OD is drawn perpendicular to BC. Prove that a circle having O as center and OD as radius is tangent to the three sides of the triangle.

5. The center O of a given circle is joined to a point P outside the circle. Another circle is described on OP as diameter, cutting the given circle in T and T'. Prove that PT and PT' are tangent to the given circle. What construction problem is solved by the method described in this exercise?

(II). *An inscribed angle is measured by one-half its intercepted arc.*



Given $\angle ACB$ inscribed in $\odot O$.

To prove $\angle ACB$ is meas. by $\frac{1}{2}\overset{\frown}{AB}$.

Proof

Case I. *One side of the angle is a diameter*

1. Draw OB.
2. $BO = OC$.
3. $\angle ACB = \angle B$.
4. $\angle AOB = \angle ACB + \angle B$.
5. $\angle AOB = 2\angle ACB$.
6. $\therefore \angle ACB = \frac{1}{2}\angle AOB$.
7. But $\angle AOB$ is meas. by $\overset{\frown}{AB}$.
8. $\therefore \angle ACB$ is meas. by $\frac{1}{2}\overset{\frown}{AB}$.

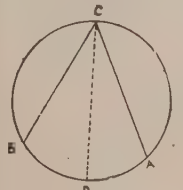
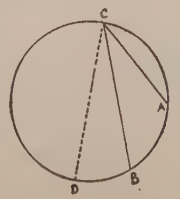


FIG. 69.

Case II. *The center lies within the angle*

1. Draw diameter CD.
2. $\angle ACD$ is meas. by $\frac{1}{2}\overset{\frown}{AD}$.
3. $\angle BCD$ is meas. by $\frac{1}{2}\overset{\frown}{BD}$.
4. $\therefore \angle ACB$ is meas. by $\frac{1}{2}\overset{\frown}{AB}$.

Case III. *The center lies without the angle*

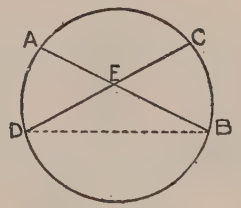


1. Draw diameter CD.
2. $\angle ACD$ is meas. by $\frac{1}{2}\overset{\frown}{AD}$.
3. $\angle BCD$ is meas. by $\frac{1}{2}\overset{\frown}{BD}$.
4. $\therefore \angle ACB$ is meas. by $\frac{1}{2}\overset{\frown}{AB}$.

Cor. 1. *Angles inscribed in the same segment are equal.*

Cor. 2. *An angle inscribed in a semicircle is a right angle.*

(II). *An angle formed by two chords intersecting within a circle is measured by one-half the sum of its intercepted arc and the arc of its vertical angle.*



Given chords AB and CD intersecting within the circle at E.

To prove $\angle AED$ is meas. by $\frac{1}{2}(\overset{\frown}{AD}+\overset{\frown}{CB})$.

Proof

1. Draw DB.
2. $\angle AED = \angle B + \angle D$.
3. $\angle B$ is meas. by $\frac{1}{2}\overset{\frown}{AD}$, and $\angle D$ is meas. by $\frac{1}{2}\overset{\frown}{BC}$.
4. $\therefore \angle B + \angle D$ is meas. by $\frac{1}{2}(\overset{\frown}{AD}+\overset{\frown}{CB})$.
5. $\therefore \angle AED$ is meas. by $\frac{1}{2}(\overset{\frown}{AD}+\overset{\frown}{CB})$.

MEASURE OF ANGLES

EXERCISE 27

1. Three consecutive sides of an inscribed quadrilateral subtend arcs of 70°, 85°, and 98° respectively. Find each angle of the quadrilateral and the angles between the diagonals.

2. Find the number of degrees in the angle formed by two secants which meet outside a circle and intercept arcs of 4/9 and 2/15 of the circle.

3. If two chords are perpendicular to each other, the sum of two opposite intercepted arcs is equal to a semicircle.

4. Prove that the line joining the points indicating 6 and 9 on the clock face is perpendicular to the line joining the points indicating 2 and 7.

5. Two secants meet at an angle of 44°. The smaller intercepted arc is 71°. How many degrees are there in the greater intercepted arc?

6. Two tangents to a circle form an angle of 75°. Find the number of degrees in each intercepted arc.

7. How many degrees are there between the hands of a clock at seven minutes past four?

8. Two intersecting chords divide a circle into arcs whose lengths, taken in order, are 3, 4, 7, and 10 inches. How many degrees are there in the angles made by the chords?

9. AB and AC are equal chords including an angle of 50°. Through B a tangent BD is drawn. How many degrees are there in the angle ABD? (*Two solutions.*)

10. An angle inscribed in a circle and an angle formed by two tangents drawn from an exterior point to the circle intercept the same arc. The inscribed angle is 48°; find the angle formed by the tangents.

11. ABC is a triangle inscribed in a circle whose center is O, and OD is drawn perpendicular to AC. Prove that $\angle AOD = \angle B$.

12. AB is the diameter of a circle and CD is a chord perpendicular to AB. E is any point in the arc BC, and AE cuts CD at F. Prove that $\angle EFD = \angle ACE$.

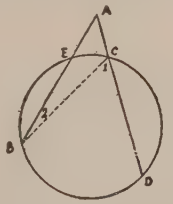
13. If from a point A in a circle a chord AB and a diameter AC are drawn, prove that the radius which bisects arc BC is parallel to AB.

14. An equiangular decagon is inscribed in a circle. Find the number of degrees in each of its angles.

15. How many degrees are there in an angle inscribed in a segment whose arc is 140°?

16. Points A, B, C, and D are taken on a circle so that arcs AB, BC, CD, and DA are equal. Prove that $\angle ABC = \angle BCD$.

(II). *An angle formed by two secants intersecting without a circle is measured by one-half the difference of the intercepted arcs.*



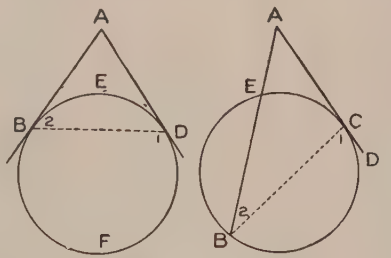
Given secants AB and AD intersecting at A.

To prove $\angle A$ is meas. by $\frac{1}{2}(\overset{\frown}{BD}-\overset{\frown}{EC})$.

Proof

1. Draw BC.
2. $\angle 1=\angle A+\angle 2$.
3. $\therefore \angle A=\angle 1-\angle 2$.
4. $\angle 1$ is meas. by $\frac{1}{2}\overset{\frown}{BD}$.
5. $\angle 2$ is meas. by $\frac{1}{2}\overset{\frown}{EC}$.
6. $\therefore \angle 1-\angle 2$ is meas. by $\frac{1}{2}(\overset{\frown}{BD}-\overset{\frown}{EC})$.
7. $\therefore \angle A$ is meas. by $\frac{1}{2}(\overset{\frown}{BD}-\overset{\frown}{EC})$.

(II). *An angle formed by two tangents or by a tangent and a secant, intersecting without a circle, is measured by one-half the difference of the intercepted arcs.*



Given tangents AB and AD intersecting at A.

To prove $\angle A$ is meas. by $\frac{1}{2}(\overset{\frown}{BFD}-\overset{\frown}{BED})$.

Proof

(*Same as in the preceding proposition*)

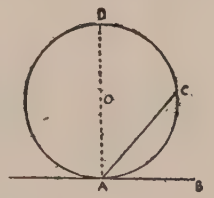
Given tangent AD and secant AB intersecting at A.

To prove $\angle A$ is meas. by $\frac{1}{2}(\overset{\frown}{BC}-\overset{\frown}{EC})$.

Proof

(*Same as in the preceding proposition*)

(II). *An angle formed by a tangent to a circle and a chord drawn from the point of contact is measured by one-half the intercepted arc.*



Given AB tangent to $\odot O$ at A, and chord AC.

To prove $\angle CAB$ is meas. by $\frac{1}{2}\overset{\frown}{AC}$.

Proof

1. Draw diameter AD.
2. $\angle DAB$ is a rt. $\angle$.
3. $\overset{\frown}{DCA}$ = a semicircumference.
4. $\therefore \angle DAB$ is meas. by $\frac{1}{2}\overset{\frown}{DCA}$.
5. But $\angle DAC$ is meas. by $\frac{1}{2}\overset{\frown}{DC}$.
6. $\therefore \angle CAB$ is meas. by $\frac{1}{2}\overset{\frown}{AC}$.

MEASURE OF ANGLES—*Concluded*

EXERCISE 28

1. Through T, the point of tangency of two circles, draw a line AB terminating in the circles. Let P and Q be any two points on the major arcs TA and TB respectively. Prove $\angle TPA = \angle TQB$.

2. If two circles are tangent externally, and through the point of contact two lines are drawn meeting one circle in B and D, and the other in E and C, BD and EC are parallel.

3. If two equal chords of a circle intersect, their corresponding segments are equal.

4. AB and CD are perpendicular diameters of a circle. K is any point on the arc ACB. Prove that DK bisects $\angle AKB$.

5. If from any point in a circle a chord and a tangent are drawn, the perpendiculars to them from the middle point of the smaller arc are equal.

6. AB and CD are two chords intersecting at E, and $CE = BE$. Prove $\overset{\frown}{AC} = \overset{\frown}{BD}$.

7. The line bisecting the angle formed by a tangent and a chord bisects the arc subtended by the chord.

8. A, B, and C are three points on a circle. The bisector of $\angle ABC$ meets the circle again at D. DE is drawn parallel to AB, and meets the circle again at E. Prove that $\overset{\frown}{DE} = \overset{\frown}{BC}$.

9. If two circles touch each other, and through the point of tangency a line is drawn terminating in the two circles, the tangents drawn at the ends of this line are parallel.

10. If from a point A outside a circle, a secant ABC and a tangent AD are drawn, and the bisector of $\angle BDC$ meets BC at E, prove that $AE = AD$.

11. Triangle ABC is inscribed in a circle. M is the midpoint of arc AC. Prove that $\angle ABC = 2\angle CBM$.

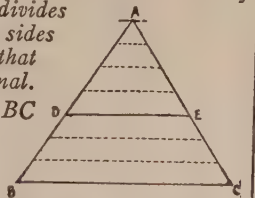
12. Two chords intersect within a circle, forming an angle of 78°. If one of the intercepted arcs is 102°, find the number of degrees in the other arc.

13. Two angles of a triangle inscribed in a circle are 58° and 62°, respectively. Find the number of degrees in each angle formed by a side of the triangle and the tangent drawn at a vertex of the triangle.

14. A chord AB is prolonged to D, and chord BC is drawn. Prove that $\angle DBC$ is measured by one-half the sum of arcs AB and BC.

15. The sides of an inscribed quadrilateral $ABCD$ cut off arcs which are in the continued ratio of 2:3:4:6. Find the number of degrees in each angle of the quadrilateral.

(III). *A line parallel to one side of a triangle divides the other two sides into segments that are proportional.*



Given $\triangle ABC$ and $DE \parallel BC$.

To prove $AD : DB = AE : EC$.

Proof

Case I. When AD and DB have a common unit of measure

1. Select a common unit of measure. Let it be contained m times in AD, and n times in DB.

2. Then $\frac{AD}{DB} = \frac{m}{n}$.

3. Lay off this unit of measure on AD and on DB as many times as it will be contained, and through the points of division and through A, draw parallels to BC.

4. Then AE will be divided into m parts, and EC into n parts all of which are equal.

5. Hence, $\frac{AE}{EC} = \frac{m}{n}$.

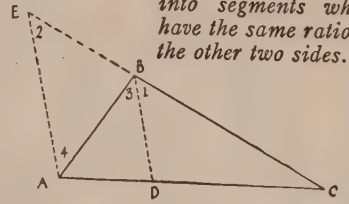
6. $\therefore \frac{AD}{DB} = \frac{AE}{EC}$.

Case II.

This proof depends on the theory of limits which is beyond the scope of this book.

Cor. *If a line parallel to one side of a triangle intersects the other two sides, one side is to either segment as the other side is to the corresponding segment.*

(III). *The bisector of an angle of a triangle divides the opposite side into segments which have the same ratio as the other two sides.*



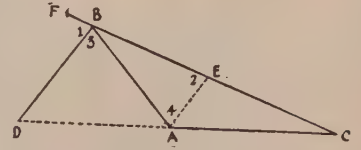
Given $\triangle ABC$, and BD bisecting $\angle ABC$.

To prove $AD : DC = AB : BC$.

Proof

1. Draw $AE \parallel BD$, meeting CB produced in E.
2. $\angle 2 = \angle 1$.
3. $\angle 1 = \angle 3$.
4. $\angle 3 = \angle 4$.
5. $\therefore \angle 2 = \angle 4$.
6. $\therefore AB = BE$.
7. $AD : DC = EB : BC$.
8. $\therefore AD : DC = AB : BC$.

(III). *The bisector of an exterior angle of a triangle divides the opposite side externally into segments which have the same ratio as the other two sides.*



Given $\triangle ABC$, and BD bisecting ext. $\angle ABF$.

To prove $AD : CD = AB : BC$.

Proof

1. Draw $AE \parallel BD$, meeting CB in E.
2. $\angle 2 = \angle 1$.
3. $\angle 1 = \angle 3$.
4. $\angle 3 = \angle 4$.
5. $\therefore \angle 2 = \angle 4$.
6. $\therefore AB = BE$.
7. $AD : CD = BE : BC$.
8. $\therefore AD : CD = AB : BC$.

PROPORTIONAL LINES

EXERCISE 29

1. In a triangle ABC, $AB = 7$ feet; $AC = 5$ feet, and $BC = 6$ feet. Find the segments of BA made by the bisector of $\angle C$. (*Hint.*—Let $AD = x$; then $BD = 7 - x$.)

2. The sides of a triangle are 9, 12, and 18. Compute the lengths of the two segments into which the longest side is divided by the bisector of the opposite angle.

3. The sides of a triangle are respectively 4, 9, and 12. In which direction, and how far, must the shortest side be produced to meet the bisector of the exterior angle at the opposite vertex?

4. BD is the bisector of $\angle B$ in $\triangle ABC$. The segments of AC are: $AD = 5$ in., and $DC = 2$ in. The sum of the sides AB and BC is 14 in. Find the lengths of AB and BC.

5. A line AB is divided into two segments by a point P. What are the segments of the line when (a) P lies between A and B? (b) When P lies on the prolongation of AB?

6. The sides of a triangle are 4, 5, and 6 yards respectively. Find the lengths of the segments into which the bisector of the angle exterior to the largest angle of the triangle divides the opposite side externally.

7. In triangle ABC, the bisectors of the interior and exterior angles at A meet the opposite side and its prolongation in D and E respectively. Prove that $BD : DC = BE : CE$.

8. Draw a line from a vertex of a triangle to the opposite side so as to divide the opposite side into segments which are proportional to the other two sides.

9. If a line drawn from a vertex of a triangle to the opposite side divides that side into segments proportional to the other two sides, the line bisects the angle from which it is drawn.

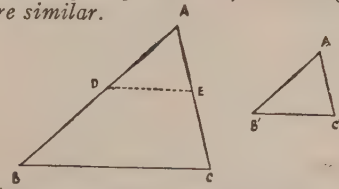
10. One side of a triangle is 21. The segments of another side made by the bisector of the opposite angle are 15 and 10. Find the third side. (*Two solutions.*)

11. In triangle ABC, $AB = 10$ in. and $AC = 8$ in. D is taken on AB so that $AD = 3$ in., and through D a line is drawn parallel to BC, cutting AC in E. Find the lengths of AE and EC.

12. One side of a triangle is 15 in. Through the point of intersection of the medians, a line is drawn parallel to another side. Find the segments into which the first side is divided.

13. Through a point within an angle draw a line ending in the sides and bisected at the point.

(III). *If two triangles have three angles of one equal respectively to three angles of the other, the triangles are similar.*



Given $\triangle ABC$ and $A'B'C'$, with $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$.

To prove $\triangle ABC \sim \triangle A'B'C'$.

Proof

1. Place $\triangle A'B'C'$ in the position of $\triangle ADE$, $\angle A'$ coinciding with its equal $\angle A$.

2. $\angle ADE = \angle B$. **3.** $\therefore DE \parallel BC$.

4. $AB : AD = AC : AE$.

5. $\therefore AB : A'B' = AC : A'C'$.

6. In the same way, by placing $\angle B'$ on $\angle B$, we can prove that $AB : A'B' = BC : B'C'$.

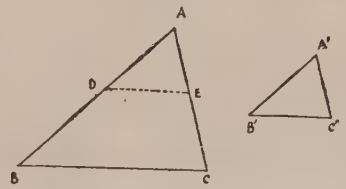
7. $\therefore AB : A'B' = AC : A'C' = BC : B'C'$.

8. $\angle A = \angle A', \angle B = \angle B', \angle C = \angle C'$.

9. $\therefore \triangle ABC \sim \triangle A'B'C'$.

Cor. *Two triangles are similar if two angles of one are equal respectively to two angles of the other.*

(III). *If two triangles have an angle of one equal to an angle of the other, and the including sides are proportional, the triangles are similar.*



Given in $\triangle ABC$ and $A'B'C'$, $\angle A = \angle A'$, and $AB : A'B' = AC : A'C'$.

To prove $\triangle ABC \sim \triangle A'B'C'$.

Proof

1. Place $\triangle A'B'C'$ in the position of $\triangle ADE$, $\angle A'$ coinciding with its equal $\angle A$.

2. $AB : A'B' = AC : A'C'$.

3. $\therefore AB : AD = AC : AE$.

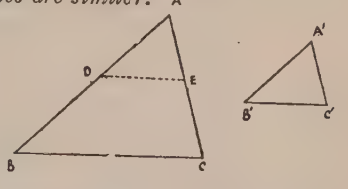
4. $DE \parallel BC$.

5. $\angle ADE = \angle B$, and $\angle AED = \angle C$,

6. $\therefore \triangle ABC \sim \triangle ADE$.

7. *i.e.*, $\triangle ABC \sim \triangle A'B'C'$.

(III). *If two triangles have their corresponding sides proportional, the triangles are similar.*



Given in $\triangle ABC$ and $A'B'C'$, $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{AC}{A'C'}$.

To prove $\triangle ABC \sim \triangle A'B'C'$.

Proof

1. On AB, lay off $AD = A'B'$, and on AC lay off $AE = A'C'$. Draw ED.

2. $\frac{AB}{A'B'} = \frac{AC}{A'C'}$. **3.** $\therefore \frac{AB}{AD} = \frac{AC}{AE}$.

4. $\angle A = \angle A$.

5. $\therefore \triangle ADE \sim \triangle ABC$.

6. $AB : AD = BC : ED$.

7. But $AB : A'B' = BC : B'C'$.

8. $\therefore ED = B'C'$.

9. $\triangle A'B'C' \cong \triangle ADE$,

10. $\therefore \triangle ABC \sim \triangle A'B'C'$.

SIMILAR TRIANGLES

SUMMARY

Triangles can be proved similar by proving—

1. Two angles of one equal to two angles of the other.
2. An angle of one equal to an angle of the other, and the including sides proportional.
3. The sides proportional. (Three equal ratios.)
4. The sides of one respectively parallel or respectively perpendicular to the sides of the other.

EXERCISE 30

1. A tangent BD and a secant AD are drawn to a circle whose diameter is AB. The secant cuts the circle at E. Prove that triangles AEB and ADB are similar.

2. AB is an arc of a circle and C is the midpoint of this arc. Through C a line is drawn, meeting the chord AB at D, and the circle again at E. Prove triangles ACD and ACE similar. Write one of the proportions that holds between the sides of these triangles.

3. A circle is drawn with one side of a triangle as a diameter, and cutting the other two sides. If the points of intersection are joined, the triangle cut off is similar to the given triangle.

4. On one side of a given angle whose vertex is O, two points P and P' are given, and on the other side two points Q and Q' are given so that $OP \times OP' = OQ \times OQ'$ Prove that the triangles OPQ and $OP'Q'$ are similar.

5. Construct a triangle similar to a given triangle and having its perimeter equal to a given line.

6. If the middle points of the sides of a triangle are joined, the four triangles formed are similar to the given triangle.

7. $ABCD$ is a quadrilateral inscribed in a circle. The sides AB and DC are produced to meet at E. Prove that triangles ACE and BDE are similar.

8. Two isosceles triangles are similar if they have a base angle of one equal to a base angle of the other.

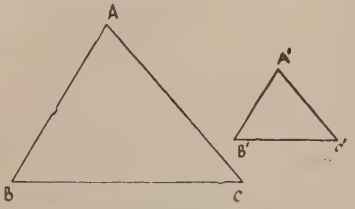
9. Prove that two isosceles triangles are similar if the vertex angle of one is equal to the vertex angle of the other.

10. Triangle ABC is inscribed in a circle, and diameter AD is drawn. The tangent at D meets AB produced at E, and AC produced at F. Prove that $\triangle ABC$ and AEF are similar.

11. In $\triangle ABC$, altitudes AD and BE are drawn. Prove that (a) $\triangle BEC$ and ADC are similar; (b) $\triangle ABC$ and DEC are similar.

12. AB is the hypotenuse of rt. $\triangle ABC$. Perpendiculars are drawn to AB at A and B, meeting BC produced at E, and AC produced at D. Prove that $\triangle ACE$ and BCD are similar to each other and to the given triangle.

(III). *Two triangles are similar if the sides of one are respectively parallel to the sides of the other.*



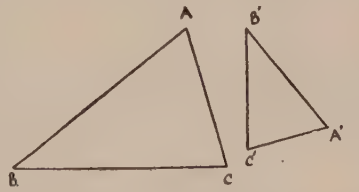
Given in $\triangle ABC$ and $A'B'C'$, $AB \parallel A'B'$, $BC \parallel B'C'$, and $AC \parallel A'C'$.

To prove $\triangle ABC \sim \triangle A'B'C'$.

Proof

1. $\angle A$ and $\angle A'$ are either equal or supplementary.
2. The same is true for $\angle B$ and $\angle B'$, and $\angle C$ and $\angle C'$.
3. $\therefore$ *Either* (a) all three pairs of angles are supplementary, *or* (b) two pairs are supplementary, and one pair equal, *or* (c) two pairs are equal.
4. If (a) or (b) were true, the sum of the angles of the two triangles would exceed 4 rt. $\angle$s. **5.** This is impossible.
6. $\therefore$ Two pairs of angles are equal.
7. $\therefore \triangle ABC \sim \triangle A'B'C'$.

(III). *Two triangles are similar if the sides of one are perpendicular respectively to the sides of the other.*



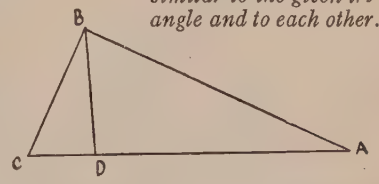
Given in $\triangle ABC$ and $A'B'C'$, $AB \perp A'B'$, $BC \perp B'C'$, and $AC \perp A'C'$.

To prove $\triangle ABC \sim \triangle A'B'C'$.

Proof

(*Exactly like proof of preceding proposition*)

(III). *In any right triangle, the altitude on the hypotenuse divides the triangle into two triangles which are similar to the given triangle and to each other.*



Given rt. $\triangle ABC$, and BD the altitude to the hypotenuse.

To prove $\triangle ABD \sim \triangle ABC$, $\triangle DBC \sim \triangle ABC$, and $\triangle ABD \sim \triangle DBC$.

Proof

1. In $\triangle ABD$ and ABC, $\angle ADB = \angle ABC$.
2. $\angle A = \angle A$.
3. $\therefore \triangle ABD \sim \triangle ABC$.
4. In $\triangle DBC$ and ABC, $\angle BDC = \angle ABC$.
5. $\angle C = \angle C$.
6. $\therefore \triangle DBC \sim \triangle ABC$.
7. In $\triangle ABD$ and DBC, $\angle ADB = \angle BDC$.
8. $\angle ABD = \angle DCB$.
9. $\therefore \triangle ABD \sim \triangle DBC$.

SIMILAR TRIANGLES—*Concluded*

EXERCISE 31

1. Using a given line as side, construct a triangle similar to a given triangle.

2. The bisectors of the exterior angles of an equilateral triangle form a triangle that is also equilateral.

3. Two sides AD and BC of an inscribed quadrilateral are produced till they meet in E. Prove that triangles ABE and CDE are similar.

4. Two circles O and O' intersect at B and C. Chord BA in circle O is tangent to circle O', and chord BD in circle O' is tangent to circle O. Prove triangles ABC and BCD similar.

5. Two triangles are similar if two sides and the median drawn to one of these sides in one triangle are proportional to two sides and the corresponding median in the other triangle.

6. On a given line as side, construct a quadrilateral similar to a given quadrilateral.

7. Two parallelograms are similar if an angle of one is equal to an angle of the other and the including sides are proportional.

8. Construct a parallelogram similar to a given parallelogram and having a given diagonal.

9. Inscribe in a given circle a triangle similar to a given triangle.

10. Quadrilateral $ABCD$ is inscribed in a circle. Its diagonals intersect at E. Prove (a) triangles AEB and DEC are similar; (b) triangles AED and BEC are similar.

11. Two circles intersect at A and B. Through A two lines are drawn, meeting the circles in C, D, and E, F, respectively. Prove that triangles BCD and BEF are similar.

12. If perpendiculars are drawn at the extremities of the hypotenuse of a right triangle to meet the legs produced, any two triangles in the figure are similar.

13. Prove that any two right isosceles triangles are similar.

14. In similar $\triangle ABC$ and $A'B'C'$, AD and $A'D'$ are altitudes and AF and $A'F'$ are angle-bisectors. Prove that $\triangle ADF$ and $A'D'F'$ are similar.

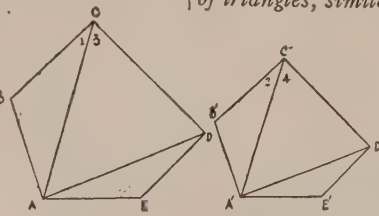
15. In isosceles triangle ABC, the bisectors of the equal angles meet the equal sides at points D and E respectively; prove that $\triangle ADE$ is similar to $\triangle ABC$.

16. Through the midpoint D of arm AB of isosceles $\triangle ABC$, a perpendicular is drawn, meeting the base (produced if necessary) in E. Prove that $\triangle ABC$ and ABE are similar.

17. FG is tangent to a circle at point A and HK is a chord parallel to FG. Chords AB and AC intersect HK at D and E, respectively. Prove that $\triangle ADE$ and ABC are similar.

18. *Prove:* two isosceles triangles are similar if an arm and the base of one are proportional respectively to an arm and the base of the other.

(III). *Two similar polygons may be divided into the same number of triangles, similar each to each, and similarly placed.*



Given polygon $ABCDE \sim$ polygon $A'B'C'D'E'$.

To prove $\triangle ABC \sim \triangle A'B'C'$, $\triangle ACD \sim \triangle A'C'D'$, etc.

Proof

1. $\frac{AB}{A'B'} = \frac{BC}{B'C'}$. **2.** $\angle B = \angle B'$.

3. $\therefore \triangle ABC \sim \triangle A'B'C'$.

4. $\angle BCD = \angle B'C'D'$.

5. $\angle 1 = \angle 2$.

6. $\therefore \angle 3 = \angle 4$. **7.** $\frac{BC}{B'C'} = \frac{AC}{A'C'}$.

8. $\frac{BC}{B'C'} = \frac{CD}{C'D'}$. **9.** $\frac{AC}{A'C'} = \frac{CD}{C'D'}$.

10. $\therefore \triangle ACD < \triangle A'C'D'$.

11. In the same way, we can prove that $\triangle ADE$ and $A'D'E'$ are similar.

(III). *Two polygons are similar if they are composed of the same number of triangles, similar each to each, and similarly placed.*

Given in polygons $ABCDE$ and $A'B'C'D'E'$, $\triangle ABC \sim \triangle A'B'C'$, $\triangle ACD \sim \triangle A'C'D'$, and $\triangle ADE \sim \triangle A'D'E'$.

To prove polygon $ABCDE \sim$ polygon $A'B'C'D'E'$.

Proof

1. $\angle B = \angle B'$, and $\angle E = \angle E'$.

2. $\angle 1 = \angle 2$, and $\angle 3 = \angle 4$.

3. $\therefore \angle BCD = \angle B'C'D'$.

4. Similarly, $\angle CDE = \angle C'D'E'$, and $\angle BAE = \angle B'A'E'$.

5. $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'}$.

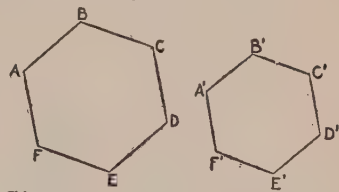
6. $\frac{CA}{C'A'} = \frac{CD}{C'D'} = \frac{DA}{D'A'}$.

7. $\frac{DA}{D'A'} = \frac{DE}{D'E'} = \frac{EA}{E'A'}$.

8. $\therefore \frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \frac{DE}{D'E'} = \frac{EA}{E'A'}$.

9. $\therefore$ polygon $ABCDE \sim$ polygon $A'B'C'D'E'$.

(V). *Regular polygons having the same number of sides are similar.*



Given regular polygons $ABCDEF$ and $A'B'C'D'E'F'$, having the same number of sides.

To prove polygon $ABCDEF \sim$ polygon $A'B'C'D'E'F'$.

Proof

1. Let n represent the number of sides in each polygon.

2. Each angle of the polygons is equal to $n-2/n$ st. $\measuredangle$.

3. $\therefore$ the polygons are mutually equiangular.

4. $AB = BC = CD = DE = EF = FA$.

5. $A'B' = B'C' = C'D' = D'E' = E'F' = F'A'$.

6. $\therefore \frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \frac{DE}{D'E'} = \frac{EF}{E'F'} = \frac{FA}{F'A'}$.

7. $\therefore$ polygon $ABCDEF \sim$ polygon $A'B'C'D'E'F'$.

SIMILAR POLYGONS

Proportions

SUMMARY

1. Corresponding sides of similar triangles are in proportion.

Note.—In applying this, select two triangles such that each has as sides two of the four lines in the required proportion, and prove these triangles similar. Mark the three pairs of equal angles and take as corresponding sides those lying opposite equal angles. If such triangles cannot be found or constructed, prove each of the required ratios equal to some third ratio.

2. In some cases, a proportion can be proved by using the following axiom: *If equals are divided by equals, the quotients are equal.*

EXERCISE 32

1. The diagonals of a trapezoid divide each other proportionally.

2. The bisector of $\angle C$ of the inscribed triangle ABC cuts AB in D and the circle in E. Prove that $AC : CD = CE : BC$.

3. Corresponding sides of similar polygons have the same ratio as any two corresponding diagonals.

4. In the parallelogram $ABCD$, BF is drawn, cutting the diagonal AC in E, CD in G, and AD prolonged in F. Prove that BE is the mean proportional between GE and EF. (*Hint.*—Use $\frac{AE}{EC}$ as a third ratio.)

5. Two isosceles triangles are similar if an angle of one is equal to an angle of the other.

6. In similar triangles, corresponding medians have the same ratio as any two corresponding sides.

7. In similar triangles, corresponding angle-bisectors have the same ratio as any two corresponding sides.

8. The perpendiculars from any two vertices of a triangle to the opposite sides are inversely proportional to those sides. (*Note.*—If four numbers, a, a', b, b' are inversely proportional, then $a : a' = b' : b$.)

9. In triangle ABC, $\angle B$ is a right angle, and BD is perpendicular to AC. Prove that $CD : BD = BD : DA$.

10. If two circles are tangent externally, and through the point of contact a secant is drawn, the chords formed are proportional to the radii.

11. If C is the midpoint of arc AB, and chord CD meets the chord AB in E, then $CE : CA = CA : CD$.

12. If the angle bisector CD of the inscribed triangle ABC is

produced to meet the circle in E, then $EA : EC = DB : CB$. (*Hint.*—EA will equal EB.)

13. Triangle ABC is inscribed in a circle. AE is a diameter and AD is the altitude to side BC. Prove that $BD : EC = AD : AC$.

14. In $\triangle ABC$, altitudes AD and BE are drawn. Prove $AD : BE = AC : BC$.

15. Two circles are tangent at T, and through T two secants ATB and CTD are drawn. Prove $AC : BD = AT : TB = CT : TD$.

16. The line of centers of two non-intersecting circles divides a common internal tangent into segments which are to each other as the diameters.

17. If in $\triangle ABC$ the altitudes BD and AE meet in F, and $AB = BC$, then $BC : AF = BD : CD$. (*Hint.*—CD will be equal to AD.)

18. If two circles intersect in A and B, and the two chords BD and BC are tangent to the two circles respectively, then AB is a mean proportional between AD and AC.

The Product of Two Lines Equal to the Product of Two Other Lines

Note.—To prove the product of two lines equal to the product of two other lines, obtain from similar triangles a proportion containing the lines, and place the product of the means equal to the product of the extremes.

EXERCISE 33

1. In a circle whose center is O, AOB is a diameter, CD a chord perpendicular to AB and crossing it at point N, and AE a chord crossing CN at F. Prove $AB \times AN = AE \times AF$.

2. Tangents are drawn to a circle at the extremities of a diameter AB. Secants are drawn from B and A, meeting the tangents at D and E respectively, and intersecting at C on the circle. Prove that $\overline{AB}^2 = AD \times BE$.

3. If two tangents are drawn to a circle at the extremities of a diameter, the portion of a third tangent intercepted between them is divided at the point of contact into segments whose product is equal to the square of the radius.

4. The product of the two arms of a right triangle is equal to the product of the hypotenuse and the altitude on the hypotenuse.

5. In triangle ABC, altitudes AD and BE meet in F. Prove $BD \times DC = DF \times AD$.

6. AB is a diameter and AT a tangent to a circle. C and D are any two points in the tangent. CB and DB meet the circle in E and F respectively. Prove that $BE \times BC = BF \times BD$.

7. From point P two secants are drawn cutting a circle in A, B, and C, D, respectively. $\angle BAD = \angle CAP$. Prove that $BD \times AP = CP \times AD$.

8. In isosceles triangle ABC, a perpendicular DE is drawn at the midpoint of the arm AB, meeting the base AC at E. Prove that $AC \times AE = \overline{AB}^2$.

9. The product of any altitude of a triangle and its corresponding side is equal to the product of any other altitude and its corresponding side.

NUMERICAL PROPORTIONS

EXERCISE 34

1. The sides of a triangle are 7, 6, and 10 respectively. In a similar triangle, the side corresponding to 7 is 9. Find the other sides of the second triangle.

2. The sides of a polygon are 2, 5, 7, 4, and 3 respectively. Find the sides and perimeter of a similar polygon whose side corresponding to 7 is 6.

3. The sides of a triangle are 8, 10, and 12 in. respectively. A line 9 in. long, parallel to the longest side, terminates in the other two sides. Find the lengths of the segments into which it divides them.

4. The centers of two circles whose radii are 12 in. and 8 in. respectively are 28 in. apart. A common external tangent is drawn and produced to meet the line of centers. How far is it from the center of the smaller circle to the intersection point of the tangent and the line of centers?

5. The arms of a right triangle are 12 in. and 16 in. respectively. Find the length of a line drawn parallel to the 16 in. side, 5 in. from it, and terminating in the other two sides.

6. The sides of a triangle are 10, 16, and 25 ft. respectively. A similar triangle has one side 6 ft. in length. Find the other sides. (*Three solutions.*)

7. The sides of a triangle are 6 in., 8 in., and 12 in., respectively. The sides of a second triangle are 6 in., 3 in., and 4 in., respectively. Are the triangles similar?

8. The sides of a triangle are 30, 30, and 48 respectively. The altitude to side 48 is 18. Find the other altitudes.

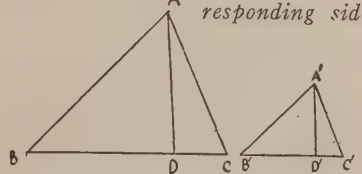
9. The sides of a triangle are 60, 50, and 72 respectively. A line is drawn joining the trisection points of sides 60 and 72, which are farthest from side 50. Find the length of this line.

10. In drawing a map of a triangular field, with sides 50 rods, 60 rods, and 40 rods respectively, the shortest side is drawn 1 inch long. How long should the other two sides be drawn?

11. The perimeter of an equilateral triangle is 32 inches. Find the side of an equilateral triangle whose altitude is one-fourth that of the given triangle.

12. The bases of a trapezoid are 8 in. and 12 in. respectively, and the altitude is 6 in. Find the altitudes of the two triangles formed by producing the non-parallel sides until they meet.

(III). *Corresponding altitudes of similar triangles have the same ratio as any two corresponding sides.*



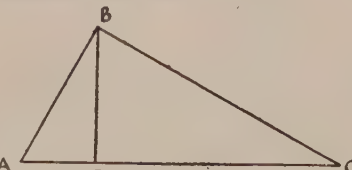
Given similar $\triangle ABC$ and $A'B'C'$, and corresponding altitudes AD and $A'D'$.

To prove $\frac{AD}{A'D'}=\frac{AB}{A'B'}=\frac{AC}{A'C'}=\frac{BC}{B'C'}$.

Proof

1. $\triangle ABC\sim\triangle A'B'C'$.
2. $\angle B=\angle B'$.
3. $\angle ADB=\angle A'D'B'$.
4. $\therefore\ \triangle ABD\sim\triangle A'B'D'$.
5. $\frac{AD}{A'D'}=\frac{AB}{A'B'}$.
6. But $\frac{AB}{A'B'}=\frac{AC}{A'C'}=\frac{BC}{B'C'}$.
7. $\therefore\ \frac{AD}{A'D'}=\frac{AB}{A'B'}=\frac{AC}{A'C'}=\frac{BC}{B'C'}$.

(III). *In a right triangle, the altitude on the hypotenuse is the mean proportional between the segments of the hypotenuse, and either arm is the mean proportional between the whole hypotenuse and the segment of it adjacent to the arm.*



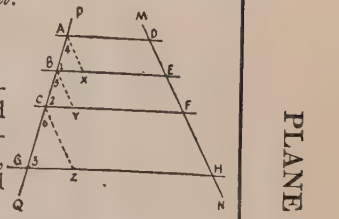
Given rt. $\triangle ABC$, and BD, the altitude on the hypotenuse.

To prove (*a*) $CD:DB=DB:DA$; (*b*) $CA:CB=CB:CD$; (*c*) $CA:AB=AB:AD$.

Proof

1. $\triangle ABD$ and CBD are similar.
2. $\therefore\ CD:DB=DB:DA$.
3. $\triangle BCD$ and ABC are similar.
4. $\therefore\ CA:CB=CB:CD$.
5. $\triangle BAD$ and ABC are similar.
6. $\therefore\ CA:AB=AB:AD$.

(III). *The segments cut off on two transversals by three or more parallels are proportional.*



Given transversals PQ and MN cut by parallel lines AD, BE, CF, and GH.

To prove $\frac{AB}{BC}=\frac{DE}{EF}$, and $\frac{BC}{CG}=\frac{EF}{FH}$.

Proof

1. Draw AX, BY, and CZ, each parallel to MN.
2. $ADEX$, $BEFY$ and $CFHZ$ are $\square$.
3. $AX=DE$, $BY=EF$, and $CZ=FH$.
4. $AX\parallel BY\parallel CZ$. 5. $\angle 4=\angle 5=\angle 6$.
6. $\angle 1=\angle 2=\angle 3$.
7. $\therefore\ \triangle ABX$, BCY, and CGZ are similar.
8. $\frac{AB}{BC}=\frac{AX}{BY}$, and $\frac{BC}{CG}=\frac{BY}{CZ}$.
9. $\therefore\ \frac{AB}{BC}=\frac{DE}{EF}$, and $\frac{BC}{CG}=\frac{EF}{FH}$.

PROPORTIONS FROM SIMILAR TRIANGLES

EXERCISE 35

1. The median to a side of a triangle bisects any line parallel to this side and ending in the other two sides.

2. The perpendicular drawn from any point in a circle to a diameter is the mean proportional between the segments of the diameter.

3. If between two parallel tangents a third tangent is drawn, the radius is the mean proportional between the segments of the third tangent. (*Hint.*—Connect the center with the intersection points of the tangents.)

4. If AB is a diameter of a circle, BC a tangent, and AC meets the circle in D, the diameter is the mean proportional between AC and AD.

5. Two sides of a triangle are 10 and 17 respectively, and the altitude on the third side is 8. In a similar triangle the side corresponding to 10 is 12. Find the altitude to the longest side of the second triangle.

6. The upper and lower bases of a trapezoid are 7 in. and 12 in. respectively, and the altitude is 6 in. If the legs of the trapezoid are produced until they meet, find the altitudes of the two triangles thus formed.

7. The bases of a trapezoid are 25 in. and 38 in. respectively. If the legs are produced until they meet, and from the point of intersection a line is drawn dividing the lower base into segments whose lengths are 28 in. and 10 in. respectively, find the lengths of the segments into which the upper base is divided.

8. One arm of a right triangle is 7 and the hypotenuse is 25. Find the lengths of the segments of the hypotenuse made by the altitude, and the length of the other arm.

9. The altitude to the hypotenuse of a right triangle is 12 and the hypotenuse is 25. Find the segments of the hypotenuse and the arms.

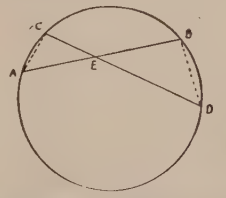
10. In two similar triangles, corresponding altitudes have the same ratio as any two corresponding medians.

11. The hypotenuse of an isosceles right triangle is 18. Find the length of an arm.

12. The square of an arm of a right triangle is equal to the product of the hypotenuse and the adjacent segment made by the altitude on the hypotenuse.

13. The squares of the arms of a right triangle are to each other as the adjacent segments of the hypotenuse made by the altitude to the hypotenuse.

(III). *If two chords intersect within a circle, the product of the segments of one is equal to the product of the segments of the other.*



Given chords AB and CD intersecting at E.

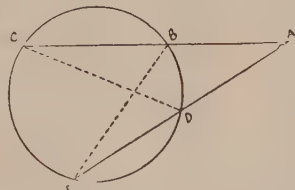
To prove $AE \times EB = CE \times ED$.

Proof

1. Draw CA and BD.
2. $\measuredangle CAB$ and $\angle CDB$ are each meas. by $\frac{1}{2}\overset{\frown}{CB}$.
3. $\therefore \angle CAB = \angle CDB$.
4. $\angle CEA = \angle BED$.
5. $\therefore \triangle CEA$ and BED are similar.
6. $\therefore AE : ED = CE : EB$.
7. *Or* $AE \times EB = CE \times ED$.

Note—This proposition may also be stated thus: *If through a fixed point within a circle a chord is drawn, the product of the segments of the chord is constant for all directions of the chord.*

(III). *If from a point outside a circle two secants are drawn, the product of one secant and its external segment is equal to the product of the other secant and its external segment.*



Given secants AC and AE drawn from external point A.

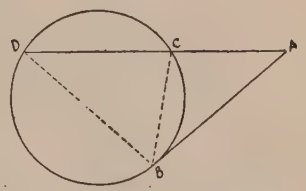
To prove $AC \times AB = AE \times AD$.

Proof

1. Draw CD and BE.
2. $\angle C$ and $\angle E$ are each meas. by $\frac{1}{2}\overset{\frown}{BD}$.
3. $\therefore \angle C = \angle E$. 4. $\angle A = \angle A$.
5. $\therefore \triangle ACD$ and ABE are similar.
6. $\therefore AC : AE = AD : AB$,
7. Or $AC \times AB = AE \times AD$.

Note—This proposition may also be stated thus: *If a secant is drawn from a point without a circle, the product of the secant and its external segment is constant for all directions of the secant.*

(III). *If from a point outside a circle, a tangent and a secant are drawn, the tangent is the mean proportional between the whole secant and its external segment.*



Given tangent AB and secant AD drawn from point A outside the circle.

To prove $AD : AB = AB : AC$.

Proof

1. Draw CB and DB.
2. $\angle D$ is meas. by $\frac{1}{2}\overset{\frown}{BC}$.
3. $\angle ABC$ is meas. by $\frac{1}{2}\overset{\frown}{BC}$.
4. $\therefore \angle D = \angle ABC$.
5. $\angle A = \angle A$.
6. $\therefore \triangle ABD \sim \triangle ABC$.
7. $\therefore AD : AB = AB : AC$.

PROPORTIONS FROM SIMILAR TRIANGLES
(Concluded)

EXERCISE 36

1. If one chord is bisected by another chord, either segment of the first chord is a mean proportional between the segments of the other.

2. The tangents to two intersecting circles from any point in their common chord produced are equal.

3. If two circles are tangent, and from any point in their common internal tangent secants are drawn to the two circles, the product of each secant and its external segment is constant.

4. The length of a secant from a certain point outside a circle is 25 in., and the length of a tangent from the same point is 15 in. Find the segments of the secant.

5. The distance from the midpoint of a chord 10 ft. long to the midpoint of its arc is 3 ft. Find the radius of the circle.

6. From a given point a tangent 24 in. long is drawn to a circle whose radius is 18 in. Find the distance of the point from the center of the circle.

7. Find the product of the segments of any chord drawn through a point 18 in. from the center of a circle whose radius is 28 in. Find also the length of the shortest chord that can be drawn through that point.

8. One of two secants meeting outside a circle is 12½ in. long. The other secant is bisected by the circle. Find the length of the second secant, if the external segment of the first is 4 in.

9. A tangent from a certain point outside a circle is 15 in. long, and the internal segment of a secant from the same point is 16 in. long. Find the external segment of the secant.

10. In a certain circle two chords whose lengths are 12 ft. and 18 ft. respectively, meet in a point which divides the longer chord into segments whose lengths are 2 ft. and 16 ft. respectively. Find the lengths of the segments of the shorter chord.

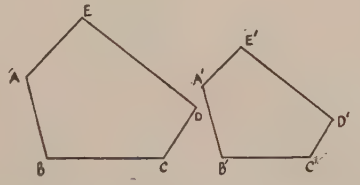
11. The radii of two concentric circles are 5 and 13 respectively. Find the length of a chord of the larger circle which is tangent to the smaller circle.

12. Through a point 21 in. from the center of a circle whose radius is 15 in. a secant is drawn. Find the product of the whole secant and its external segment.

13. If two equal secants are drawn from a point outside a circle, (a) their internal segments are equal; (b) their external segments are equal.

14. The radius of a circle is 36 ft. Find the length of a tangent drawn from a point 9 ft. from the circle.

(III). *The perimeters of two similar polygons are to each other as any two corresponding sides.*



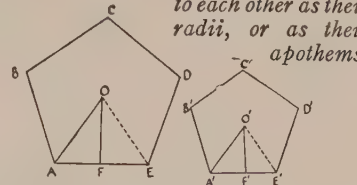
Given similar polygons $ABCDE$ and $A'B'C'D'E'$ with perimeters P and P' respectively.

To prove $P : P' = AB : A'B' = BC : B'C'$, etc.

Proof

1. $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \ldots.$
2. $\frac{AB + BC + \ldots}{A'B' + B'C' + \ldots} = \frac{AB}{A'B'} = \frac{BC}{B'C'} \cdots$
3. $\therefore \frac{P}{P'} = \frac{AB}{A'B'} = \frac{BC}{B'C'} = \ldots.$

(V). *The perimeters of regular polygons having the same number of sides are to each other as their radii, or as their apothems.*



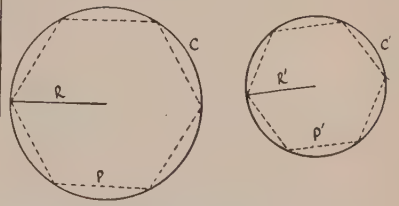
Given regular polygons $ABCDE$ and $A'B'C'D'E'$, with perimeters P and P', and having the same number of sides. OA and $O'A'$ are radii. OF and $O'F'$ are apothems.

To prove $P : P' = OA : O'A' = OF : O'F'$.

Proof

1. $ABCDE \sim A'B'C'D'E'$.
2. $\therefore \frac{P}{P'} = \frac{AE}{A'E'}$.
3. Draw OE and $O'E'$.
4. $\angle AOE = \angle A'O'E'$.
5. $OA = OE$.
6. $O'A' = O'E'$.
7. $\therefore \frac{OA}{O'A'} = \frac{OE}{O'E'}$.
8. $\triangle AOE \sim \triangle A'O'E'$.
9. $\therefore \frac{AE}{A'E'} = \frac{OA}{O'A'}$.
10. $\frac{AE}{A'E'} = \frac{OF}{O'F'}$.
11. $\therefore \frac{P}{P'} = \frac{OA}{O'A'} = \frac{OF}{O'F'}$.

(V). *Circumferences are to each other as their radii.*



Given circumferences C and C', with radii R and R'.

To prove $\frac{C}{C'} = \frac{R}{R'}$.

Proof

1. Inscribe in each circle a regular polygon of n sides. Let P and P' be their perimeters.
2. $\frac{P}{P'} = \frac{R}{R'}$.
3. *Or*, $\frac{P}{R} = \frac{P'}{R'}$.
4. Imagine n to increase indefinitely. Then, $P \to C$, and $P' \to C'$.
5. $\therefore \frac{P}{R} \to \frac{C}{R}$, and $\frac{P'}{R'} \to \frac{C'}{R'}$.
6. $\therefore \frac{C}{R} = \frac{C'}{R'}$.
7. *Or*, $\frac{C}{C'} = \frac{R}{R'}$.

PROPORTIONAL PERIMETERS

EXERCISE 37

1. The perimeters of two similar polygons have the same ratio as any two corresponding diagonals.

2. The perimeters of two similar triangles are to each other as any two corresponding altitudes, or as any two corresponding medians.

3. The perimeters of two similar polygons are 80 and 64 respectively, and a side of the first polygon is 10. Find the corresponding side of the second.

4. The perimeters of two similar polygons are 76 and 114 respectively. If a diagonal of the first is 16, what is the corresponding diagonal of the second?

5. The sides of a polygon are 5, 7, 10, 14, and 19 respectively. Find the perimeter of a similar polygon if the side corresponding to 7 is 12.

6. A regular polygon, the side of which is 6 in., is circumscribed about a circle whose radius is $\sqrt{3}$ in. Find the side of a similar polygon circumscribed about a circle whose radius is 6 in.

7. Circumferences are to each other as their diameters.

8. The circumferences of two circles are 942.48 ft. and 157.08 ft. respectively. The diameter of the first is 300 ft.; find the diameter of the second.

9. The radius of a circle is 8 in. If the radius is multiplied by 6, what will be the ratio of the two circumferences?

10. The radius of a circle is 8 in. If the radius is increased by 6 in., what will be the ratio of the two circumferences?

11. Construct a circumference whose length shall be twice that of a given circumference.

12. On the radius of a given circle as diameter another circle is constructed. What is the ratio of the circumferences of the two circles?

13. *Prove:* The perimeters of two similar triangles are to each other as any two corresponding angle-bisectors.

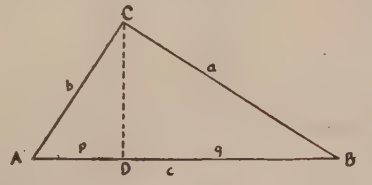
14. Construct a right triangle, having given its perimeter and an acute angle. (*Hint.*—Any right triangle containing the given angle is similar to the given triangle.

15. What is the ratio of the circumferences of two circles constructed on the diagonal and the side of a square respectively.

16. *Prove:* The perimeters of two similar triangles are to each other as the radii of the inscribed circles.

17. *Prove:* The perimeters of two similar triangles are to each other as the radii of the circumscribed circles.

(III). *The sum of the squares of the arms of a right triangle is equal to the square of the hypotenuse.*



Given rt. $\triangle ABC$ having its rt. $\angle$ at C.

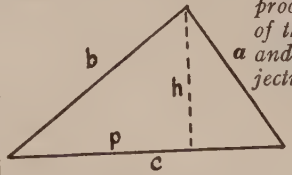
To prove $a^2+b^2=c^2$.

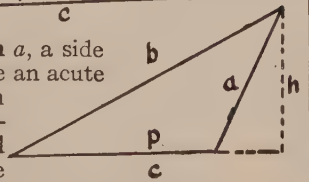
Proof

1. Draw $CD \perp AB$.
2. $p : b = b : c$.
3. $\therefore b^2 = p \times c$.
4. $q : a = a : c$.
5. $\therefore a^2 = q \times c$.
6. $a^2+b^2 = p \times c + q \times c$.
   $= c(p+q)$.
7. $\therefore a^2+b^2=c^2$.

Cor. *The square of either arm of a right triangle is equal to the square of the hypotenuse diminished by the square of the other arm.*

(III). *In any triangle, the square of a side opposite an acute angle is equal to the sum of the squares of the other two sides diminished by twice the product of one of those sides and the projection of the other side upon it.*





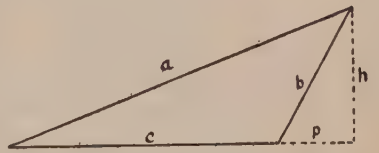
Given a, a side opposite an acute angle in a triangle, and p, the projection of b on c.

To prove $a^2=b^2+c^2-2cp$.

Proof

1. Draw h, the altitude on c.
2. In the acute triangle, $a^2=h^2+(c-p)^2$.
3. In the obtuse triangle, $a^2=h^2+(p-c)^2$.
4. $\therefore$ In either figure, $a^2=h^2+c^2-2cp+p^2$.
5. But $h^2+p^2=b^2$.
6. $\therefore a^2=b^2+c^2-2cp$.

(III). *In any obtuse triangle, the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides increased by twice the product of one of those sides and the projection of the other side upon it.*



Given a, the side opposite the obtuse angle in an obtuse triangle, and p, the projection of b on c.

To prove $a^2=b^2+c^2+2cp$.

Proof

1. $a^2=h^2+(c+p)^2$.
2. $a^2=h^2+c^2+2cp+p^2$.
3. But $h^2+p^2=b^2$,
4. $\therefore a^2=b^2+c^2+2cp$.

NUMERICAL RELATIONS BETWEEN SIDES OF TRIANGLES

EXERCISE 38

1. A chord 2 ft. long is 16 in. from the center of a circle. Find the length of the radius.

2. A chord 2 ft. long is 4 in. from the center of a circle. How far from the center is a chord 9 in. long?

3. The perimeter of a certain rectangle is 14 ft. and its diagonal is 5 ft. Find its dimensions.

4. An automobile goes north 9 miles, then east 8 miles, then north 6 miles. How far is it from the starting point at the end of the journey?

5. The shortest side of a right triangle is 30 in., and the altitude on the hypotenuse is 24 in. Find the segments of the hypotenuse and the third side of the triangle.

6. In a circle whose radius is 25 in., an arc is subtended by a chord 48 in. long. Find the length of the chord of an arc half as long.

7. Find the altitude of an equilateral triangle whose side is 12 in.

8. If the side of an equilateral triangle is a, show that the altitude is $\frac{a}{2}\sqrt{3}$.

9. Find the side of an equilateral triangle whose altitude is 10.

10. The diagonals of a rhombus are 10 and 24. Find its perimeter.

11. Determine the nature of the following triangles, as right, acute, or obtuse, if the sides are (a) 15, 17, 29; (b) 7, 21, 20; (c) 115, 35, 140; (d) 21, 28, 35.

12. Find the diagonal of a square whose side is 7.

13. A rectangle whose sides are 16 and 30 respectively is inscribed in a circle. Find the length of the radius.

14. Two poles 16 ft. apart are 18 ft. and 30 ft. high respectively. Find the distance between their tops.

15. If the sides of an acute triangle are 13, 14, and 15, find the length of the projection of side 13 on side 14.

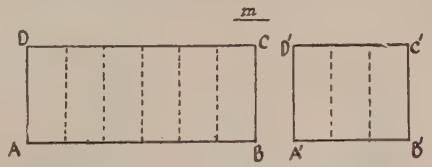
16. The sides of an obtuse triangle are 4, 13, and 15. Find the projection of side 13 on side 4.

17. In each of the triangles mentioned in Ex. 11, above, compute the length of the projection of the medium side on the shortest side.

18. Compute the length of the common external tangent and also of the common internal tangent to two circles whose radii are 8 and 6 respectively, and the distance between whose centers is 20.

19. If the sum of the squares of two sides of a triangle is equal to the square of the third side, the triangle is a right triangle.

(IV). *Rectangles having equal altitudes are to each other as their bases.*



Given rectangles $ABCD$ and $A'B'C'D'$ with equal altitudes AD and $A'D'$.

To prove $\frac{ABCD}{A'B'C'D'} = \frac{AB}{A'B'}$.

Proof

Case I. *When* $\frac{AB}{A'B'}$ *is a rational number.*

1. Let m be a common unit of measure, contained h times in AB, and k times in $A'B'$.

2. Then, $AB = hm$, and $A'B' = km$. **3.** $\therefore \frac{AB}{A'B'} = \frac{h}{k}$.

4. Lay off m on AB and on $A'B'$ as many times as it will be contained, and at the points of division erect perpendiculars.

5. $ABCD$ will be divided into h small rectangles, and $A'B'C'D'$ into k small rectangles.

6. These small rectangles are all congruent.

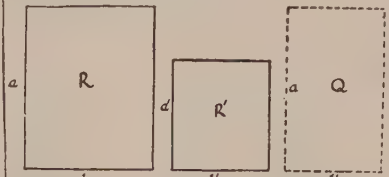
7. Let S represent the area of one of the small rectangles.

8. Then, $ABCD = hS$, and $A'B'C'D' = kS$. **9.** $\therefore \frac{ABCD}{A'B'C'D'} = \frac{h}{k}$. **10.** *or* $\frac{ABCD}{A'B'C'D'} = \frac{AB}{A'B'}$.

Case II. *When* $\frac{AB}{A'B'}$ *is not a rational number.* This proof depends on the theory of limits which is beyond the scope of this book.

Cor. *Rectangles having equal bases are to each other as their altitudes.*

(IV). *The areas of two rectangles are to each other as the products of their bases and altitudes.*



Given rectangles R and R', with bases b and b', and altitudes a and a', respectively.

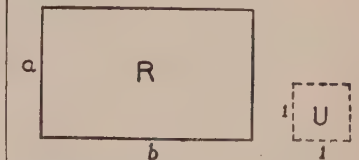
To prove $\frac{R}{R'} = \frac{a \times b}{a' \times b'}$.

Proof

1. Construct rectangle Q, having its base equal to b', and its altitude equal to a.

2. $\frac{R}{Q} = \frac{b}{b'}$. **3.** $\frac{Q}{R'} = \frac{a}{a'}$. **4.** $\therefore \frac{R}{R'} = \frac{a \times b}{a' \times b'}$.

(IV). *The area of a rectangle is equal to the product of its base and altitude.*



Given rectangle R, with base b and altitude a.

To prove area of $R = a \times b$.

Proof

1. Let U be the unit of surface.

2. $\frac{R}{U} = \frac{b \times a}{1 \times 1}$.

3. But $\frac{R}{U}$ is the area of R.

4. $\therefore$ area of $R = a \times b$.

AREAS

EXERCISE 39

1. A rectangular field whose width is 7 rods and a square field whose side is 14 rods have equal areas. Find the perimeter of each.

2. A certain rectangle is 3½ times as long as it is wide, and its area is 70 sq. in. Find its dimensions.

3. The area of a rhombus is equal to one-half the product of its diagonals.

4. The diagonals of a rhombus are 36 ft. and 40 ft. respectively. Find its area.

5. One side of a rhombus is 20 and its longer diagonal is 32. Find the other diagonal and the area.

6. The area of a trapezoid is 204 sq. in. Its altitude is 17 in. and one base is 16 in. Find the other base.

7. The parallel sides of a trapezoid are 125 ft. and 185 ft. respectively, and the other sides are each 50 ft. Find the area of the trapezoid.

8. Two adjacent sides of a parallelogram are 24 and 10 respectively and they include an angle of 60°. Find the area of the parallelogram.

9. Find the area of a regular hexagon circumscribed about a circle whose radius is 8 inches.

10. The radius of a circle is 100 ft. and its circumference is 628.32 ft. Find its area.

11. Divide a parallelogram into four equivalent parallelograms.

12. Two sides of a triangle are 12 in. and 15 in. respectively, and the included angle is 45°. Find the area.

13. Find the area of an isosceles right triangle whose hypotenuse is 20.

14. The base of a triangle is 3¼ times as long as the altitude and its area is 78 sq. in. Find the length of the base.

15. Find the area of an isosceles triangle whose base is 24 and whose arm is 15.

16. The base of a certain triangle is 18 and the altitude is 12. A line is drawn parallel to the base at a distance 5 from the base. Find the area of the trapezoid cut off by this line.

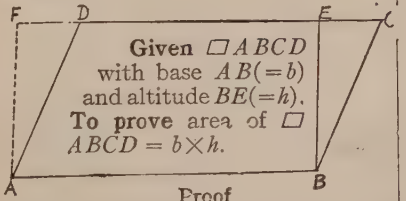
17. A rectangle, one of whose sides is 15, is inscribed in a circle whose radius is 12½. Find the area of the rectangle.

18. The area of a rectangle is 168 sq. ft. and the perimeter is 58 ft. Find its dimensions.

19. The area of a trapezoid is equal to the product of its median and altitude.

20. Find the area of a triangle whose altitude is 19.05 in. and whose base is 15.476 in.

(IV). *The area of a parallelogram is equal to the product of its base and altitude.*



Given $\square ABCD$ with base $AB(=b)$ and altitude $BE(=h)$.

To prove area of $\square ABCD = b \times h$.

Proof

1. Draw $AF \perp AB$ to meet CD produced at F. **2.** $AF \parallel BE$.

3. $ABEF$ is a parallelogram.

4. $\angle ABE$ is a rt. $\angle$.

5. $\therefore$ $ABEF$ is a rectangle, with base b, and altitude h.

6. $AD = BC$. **7.** $AF = BE$.

8. $\therefore$ Rt. $\triangle ADF \cong$ rt. $\triangle BEC$.

9. $\therefore$ Area $\square ABCD$ = area rect. $ABEF$.

10. But area of rect. $ABEF = b \times h$.

11. $\therefore$ Area $\square ABCD = b \times h$.

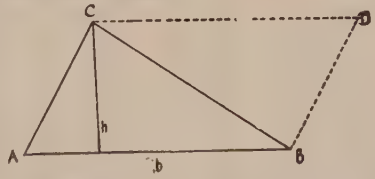
Cor. 1. *Parallelograms having equal bases and equal altitudes are equivalent.*

Cor. 2. *Any two parallelograms are to each other as the products of their bases and altitudes.*

Cor. 3. *Parallelograms having equal bases are to each other as their altitudes.*

Cor. 4. *Parallelograms having equal ltitudes are to each other as their bases.*

(IV). *The area of a triangle is equal to one-half the product of its base and altitude.*



Given $\triangle ABC$, with base b and altitude h.

To prove area $\triangle ABC = \frac{1}{2} b \times h$.

Proof

1. Draw $BD \parallel AC$, and $CD \parallel AB$.

2. $ABDC$ is a parallelogram, having base b and altitude h.

3. $\therefore$ Area $ABDC = b \times h$.

4. But $\triangle ABC = \frac{1}{2} ABCD$.

5. $\therefore$ Area $\triangle ABC = \frac{1}{2} b \times h$.

Cor. 1. *Triangles having equal bases and equal altitudes are equivalent.*

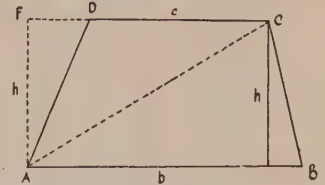
Cor. 2. *Any two triangles are to each other as the products of their bases and altitudes.*

Cor. 3. *Triangles having equal bases are to each other as their altitudes.*

Cor. 4. *Triangles having equal altitudes are to each other as their bases.*

Cor. 5. *If two triangles have a common base and their vertices lie in a line parallel to the base, the triangles are equivalent.*

(IV). *The area of a trapezoid is equal to one-half the product of its altitude and the sum of its bases.*



Given trapezoid $ABCD$ with bases b and c, and altitude h.

To prove area $ABCD = \frac{1}{2} h(b+c)$.

Proof

1. Draw AC. Draw $AF \perp CD$ produced.

2. $AF = h$.

3. In $\triangle ADC$, AF is the alt. to base CD.

4. Area $\triangle ABC = \frac{1}{2} b \times h$.

5. Area $\triangle ACD = \frac{1}{2} c \times h$.

6. $\therefore$ Area $ABCD = \frac{1}{2} b \times h + \frac{1}{2} c \times h$.

7. *or* area $ABCD = \frac{1}{2} h(b+c)$.

AREAS—*Continued*

SUMMARY

1. Triangles or parallelograms having equal bases and equal altitudes are equivalent.
2. Triangles having the same or equal bases and whose vertices lie in a line parallel to the base are equivalent.
3. A triangle is one-half a parallelogram having an equal base and an equal altitude.
4. The areas of similar figures are to each other as the squares of corresponding lines.

Note.—If none of the above applies, try one of the following:

5. Find an algebraic expression for each of the two areas and compare the two expressions with each other.
6. Divide the figures into triangles and compare the respective triangles.

EXERCISE 40

1. Show how to divide a given triangle into two equivalent triangles by means of a line drawn through a vertex.

2. The four triangles formed by the diagonals of any parallelogram are equal in area. (*Hint.*—Find two different sets of four triangles.)

3. $ABCD$ is any parallelogram and E any point within. Prove that the sum of the triangles EAD and EBC is equal to one-half the area of the parallelogram. (*Hint.*—Draw through E a line parallel to a side of the parallelogram.)

4. The area of a regular inscribed hexagon is equal to twice the area of the equilateral triangle inscribed in the same circle. (*Hint.*—Connect the center with the vertices of the triangle.)

5. The triangle formed by drawing a line from any vertex of a parallelogram to the midpoint of the opposite side is equivalent to one-fourth of the parallelogram.

6. The midpoints of two sides of a triangle are joined, and through the midpoints lines are drawn parallel to each other and meeting the third side. Prove that a parallelogram is formed whose area is one-half the area of the triangle. (Let b represent the base of the triangle, and h its altitude. Express both areas in terms of b and h.)

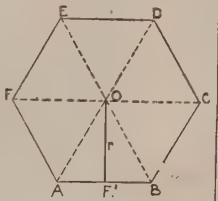
7. If the midpoints of two sides of a triangle are joined, a triangle is cut off whose area is one-fourth the area of the given triangle.

8. In triangle ABC, medians AE, CD, and BF meet in O. Prove that—

   (a) Each median divides the triangle into two equivalent triangles.

   (b) Triangles BOA, AOC, and BOC are each equivalent to one-third the given triangle. (*Hint.*—Show that the altitude of $\triangle BOA$ is one-third the altitude of $\triangle BCA$.)

(V). *The area of a regular polygon is equal to one-half the product of its perimeter and apothem.*



Given a regular polygon having n sides; P, its perimeter; S, its area; and r its apothem.

To prove $S=\frac{1}{2}P\times r$.

Proof

1. Draw a radius to each vertex of the polygon.

2. These radii divide the polygon into n triangles.

3. OA, OB, OC, etc., are all equal.

4. AB, BC, CD, etc., are all equal.

5. $\therefore$ the n triangles into which the polygon is divided are all congruent.

6. $\therefore S=n\times\triangle AOB$.

7. OF' (or r) $\perp AB$.

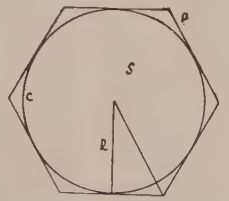
8. $\therefore\triangle AOB=\frac{1}{2}AB\times r$.

9. $\therefore S=n\times\frac{1}{2}AB\times r$ *or* $\frac{1}{2}n\times AB\times r$.

10. But $n\times AB=P$.

11. $\therefore S=\frac{1}{2}P\times r$.

(V). *The area of a circle is equal to one-half the product of its circumference and radius.*



Given a circle with circumference C, area S, and radius R.

To prove $S=\frac{1}{2}C\times R$.

Proof

1. Circumscribe a regular polygon about the circle. Let its perimeter be P, and its area, A.

2. $A=\frac{1}{2}P\times R$.

3. Let the number of sides of the polygon be increased indefinitely, then $A\rightarrow S$.

4. But $P\rightarrow C$.

5. $\frac{1}{2}P\times R\rightarrow\frac{1}{2}C\times R$.

6. $\therefore S=\frac{1}{2}C\times R$.

Note—Students taking the New York State Regents examinations are allowed to use the following assumption: *Any property of a regular polygon which is true no matter how many sides the polygon has is true also for circles.*

The proof of the preceding proposition based on this assumption is as follows:

1. As in the preceding proof.

2. As in the preceding proof.

3. Step **2** is true no matter how many sides the polygon may have.

4. $\therefore$ by the assumption,

$$S=\frac{1}{2}C\times R.$$

AREAS—*Concluded*

EXERCISE 41

1. Find the area of a regular hexagon whose side is 8 in.

2. The area of a sector is equal to one-half the product of the radius and the length of the arc.

3. Prove that two triangles are equivalent if the base of the first is equal to one-half the altitude of the second, and the altitude of the first is equal to twice the base of the second.

4. Prove that every line drawn through the point of intersection of the diagonals of a parallelogram bisects the parallelogram.

TRANSFORMATIONS

To *transform* a figure means to change its shape without changing its area. In transforming triangles it is frequently useful to draw a line through a vertex parallel to the opposite side, and making use of the fact that triangles having equal bases and equal altitudes are equivalent.

EXERCISE 42

1. Transform a given acute triangle into a right triangle.

2. Transform a given triangle into an isosceles triangle.

3. Transform a given triangle into a triangle having a given line as a side.

4. Construct a triangle equivalent to a given quadrilateral.

5. Construct a parallelogram equivalent to a given triangle.

6. Construct a parallelogram equivalent to a given trapezoid and having the same altitude as the trapezoid.

7. Transform a pentagon into a triangle.

8. Transform a given triangle into another triangle having two given lines as sides. (*Hint.*—Use two separate transformations.)

9. Transform a given triangle into another triangle containing an angle of 60°.

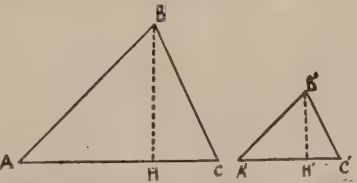
10. Construct a parallelogram equivalent to a given parallelogram *ABCD*, having the same base as the given parallelogram, and having one diagonal equal in length to a given line.

11. Construct a rhombus equivalent to a given parallelogram and having one of its diagonals coinciding with a diagonal of the parallelogram.

12. Given $\triangle ABC$ in which the median to side BC is 2 in. Construct a triangle equivalent to $\triangle ABC$, having the same base BC, and the corresponding median equal to 3 in.

13. Through any given point draw a line which shall bisect the area of a given parallelogram.

(IV). *The areas of two similar triangles are to each other as the squares of any two corresponding sides.*



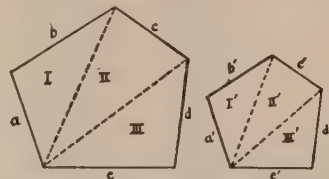
Given $\triangle ABC \sim \triangle A'B'C'$.

To prove $\frac{\triangle ABC}{\triangle A'B'C'} = \frac{\overline{AC}^2}{\overline{A'C'}^2} = \frac{\overline{AB}^2}{\overline{A'B'}^2} = \frac{\overline{BC}^2}{\overline{B'C'}^2}$.

Proof

1. Draw altitudes BH and $B'H'$.
2. $\frac{\triangle ABC}{\triangle A'B'C'} = \frac{AC \times BH}{A'C' \times B'H'} = \frac{AC}{A'C'} \times \frac{BH}{B'H'}$.
3. But $\frac{BH}{B'H'} = \frac{AC}{A'C'}$.
4. $\therefore \frac{\triangle ABC}{\triangle A'B'C'} = \frac{AC}{A'C'} \times \frac{AC}{A'C'} = \frac{\overline{AC}^2}{\overline{A'C'}^2}$.
5. But $\frac{AC}{A'C'} = \frac{AB}{A'B'} = \frac{BC}{B'C'}$.
6. And $\frac{\overline{AC}^2}{\overline{A'C'}^2} = \frac{\overline{AB}^2}{\overline{A'B'}^2} = \frac{\overline{BC}^2}{\overline{B'C'}^2}$.
7. $\therefore \frac{\triangle ABC}{\triangle A'B'C'} = \frac{\overline{AC}^2}{\overline{A'C'}^2} = \frac{\overline{AB}^2}{\overline{A'B'}^2} = \frac{\overline{BC}^2}{\overline{B'C'}^2}$.

(IV). *The areas of two similar polygons are to each other as the squares of any two corresponding sides.*



Given a and a' corresponding sides of two similar polygons whose areas are S and S' respectively.

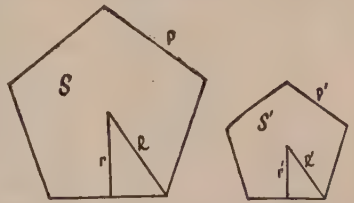
To prove $\frac{S}{S'} = \frac{a^2}{a'^2}$.

Proof

1. From any two corresponding vertices draw all possible diagonals.
2. Then, $\triangle I \sim \triangle I'$, $\triangle II \sim \triangle II'$, etc.
3. Whence $\frac{\triangle I}{\triangle I'} = \frac{a^2}{a'^2}$.
4. $\frac{\triangle II}{\triangle II'} = \frac{c^2}{c'^2}$.
5. $\frac{\triangle III}{\triangle III'} = \frac{d^2}{d'^2}$.
6. But $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$, etc.
7. $\frac{a^2}{a'^2} = \frac{b^2}{b'^2} = \frac{c^2}{c'^2}$, etc.
8. $\therefore \frac{\triangle I}{\triangle I'} = \frac{\triangle II}{\triangle II'} = \frac{\triangle III}{\triangle III'} = \frac{a^2}{a'^2}$.
9. $\therefore \frac{\triangle I + \triangle II + \triangle III}{\triangle I' + \triangle II' + \triangle III'} = \frac{a^2}{a'^2}$.
10. *i.e.*, $\frac{S}{S'} = \frac{a^2}{a'^2}$.

Cor. *The areas of two similar polygons are to each other as the squares of their perimeters, or as the squares of corresponding diagonals.*

(IV). *The areas of two regular polygons having the same number of sides are to each other as the squares of their radii or as the squares of their apothems.*



Given two regular polygons of the same number of sides, having radii R and R', apothems r and r', areas S and S', and perimeters P and P'.

To prove $\frac{S}{S'} = \frac{R^2}{R'^2} = \frac{r^2}{r'^2}$.

Proof

1. The two polygons are similar.
2. $\frac{S}{S'} = \frac{P^2}{P'^2}$.
3. But $\frac{P}{P'} = \frac{R}{R'} = \frac{r}{r'}$.
4. $\therefore \frac{P^2}{P'^2} = \frac{R^2}{R'^2} = \frac{r^2}{r'^2}$.
5. $\therefore \frac{S}{S'} = \frac{R^2}{R'^2} = \frac{r^2}{r'^2}$.

PROPORTIONAL AREAS

EXERCISE 43

1. The base of a certain triangle is 10 ft. long. Find the base of a similar triangle whose area is 2¼ times as great.

2. The corresponding altitudes of two similar triangles are 8 in. and 10 in. respectively. The area of the larger is 96 sq. in. Find the area of the second triangle and the base of each.

3. In $\triangle ABC$, D and F are the midpoints of AB and AC respectively. Prove that the area of $\triangle ADF$ is equal to one-fourth the area of $\triangle ABC$.

4. The longest sides of two similar polygons are 18 in. and 3 in. respectively. How many times as great is the area of the first than the area of the second?

5. If the area of a polygon, one of whose sides is 36 ft., is 648 sq. ft., find the corresponding side of a similar polygon whose area is 392 sq. ft.

6. The sides of a triangle are 6, 8, and 9 respectively. A line parallel to the longest side divides the triangle into a trapezoid and a triangle which are equal in area. Find the lengths of the segments into which the sides 6 and 8 are divided by the line parallel to the longest side.

7. In the triangle of the preceding exercise, find the segments of sides 6 and 8, if the line parallel to the longest side cuts off a triangle whose area is one-fourth that of the given triangle.

8. Similar triangles are to each other as the squares of any two corresponding medians or as the squares of any two corresponding angle-bisectors.

9. If one square is double another square, find the ratio of their sides.

10. The base of a triangle is 32 ft. and its altitude is 20 ft. What is the area of the triangle cut off by drawing a line parallel to the base and at a distance of 15 ft. from it.

11. In a trapezoid whose area is 48 sq. ft., one base is 6 ft. and the altitude is 4 ft. Find the length of a line drawn between the legs of the trapezoid parallel to the longer base and 1 ft. from it.

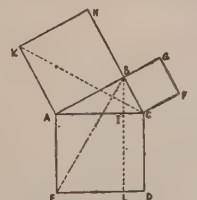
12. The bases of a trapezoid are 15 and 24 respectively. If the non-parallel sides are produced until they meet, find the ratio of the areas of the two triangles thus formed.

13. The bases of a trapezoid are 18 and 8 respectively, and its area is 78. Find the areas of the two triangles formed by prolonging the two non-parallel sides until they meet.

14. Regular pentagons are inscribed in two circles whose radii are 5 in. and 8 in. respectively. Find the ratio of the areas of the polygons.

15. A certain polygon whose perimeter is 10 ft. has an area of 25 sq. ft. Find the area of a similar polygon whose perimeter is 2½ ft.

(IV). *In a right triangle the sum of the squares on the arms is equivalent to the square on the hypotenuse.*



Given AH and BF, squares on the arms, and AD the square on the hypotenuse of the right triangle ABC.

To prove $AD \simeq AH + BF$.

Proof

1. From B draw $BL \parallel AE$, intersecting AC and ED in I and L, respectively. Draw BE and KC.

2. $\angle HBC$ = st. $\angle$.

3. HBC is a straight line.

4. In $\triangle AKC$ and BAE, $KA = AB$ and $AC = AE$.

5. $\angle KAC = \angle BAE$.

6. $\therefore \triangle KAC \cong \triangle BAE$.

7. In $\triangle AKC$, if AK be taken as the base, then AB will be equal to the altitude.

8. $\therefore$ Area $\triangle AKC = \frac{1}{2} AK \times AB$.

9. But area of square $AH = AK \times AB$.

10. $\therefore$ square $AH = 2\triangle AKC$.

11. In $\triangle BAE$, if AE be taken as the base, then AI will be equal to the altitude.

12. $\therefore$ area $\triangle BAE = \frac{1}{2} AE \times AI$.

13. But rect. $AL = AE \times AI$. **14.** $\therefore$ rect. $AL = 2\triangle BAE$.

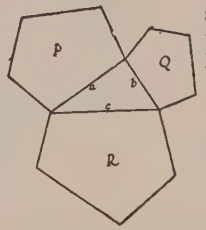
15. $\therefore$ from 6, 10, and 14, area of square $AH \simeq$ area of rectangle AL.

16. In the same way, by drawing BD and AF, we can prove square $BF \simeq$ rectangle ID.

17. $\therefore$ area $AD \simeq$ area $AH +$ area BF.

Cor. *The square on an arm of a right triangle is equivalent to the difference between the squares on the hypotenuse and the other arm.*

(IV). *If similar polygons are constructed on the three sides of a right triangle, the polygon on the hypotenuse is equivalent to the sum of the polygons on the arms.*



Given P, Q, and R, the areas of similar polygons constructed, respectively, on the arms a and b and the hypotenuse c of a rt. $\triangle$.

To prove $R \simeq P + Q$.

Proof

1. Polygons P and R are similar.

2. $\therefore \frac{P}{R} = \frac{a^2}{c^2}$.

3. Polygons Q and R are similar.

4. $\therefore \frac{Q}{R} = \frac{b^2}{c^2}$.

5. $\frac{P+Q}{R} = \frac{a^2+b^2}{c^2}$.

6. But $a^2 + b^2 = c^2$.

7. $\therefore \frac{a^2+b^2}{c^2} = \frac{c^2}{c^2} = 1$.

8. $\frac{P+Q}{R} = 1$.

9. or $P + Q = R$.

Cor. *If circles are constructed on the three sides of a right triangle the circle on the hypotenuse is equivalent to the sum of the circles on the arms.*

EQUIVALENT AREAS

EXERCISE 44

1. Construct a square equal to the sum of two given squares.

2. Construct a square equal to the difference of two given squares.

3. Construct an equilateral triangle equivalent to (a) the sum of two given equilateral triangles; (b) the difference of two given equilateral triangles.

4. Construct a circle equivalent to (a) the sum of two given circles; (b) the difference of two given circles.

5. The square described on the diagonal of a given square is double the given square.

6. The square on the hypotenuse of an isosceles right triangle is four times the area of the triangle.

7. A room is 15 ft. wide, 20 ft. long, and 12 ft. high. Find the distance from an upper corner to the opposite lower corner.

8. Find the side of a square equal in area to the sum of two squares whose sides are 186 and 248 respectively.

9. Find the diameter of a circle equivalent to the sum of two circles whose radii are 88 and 165 respectively.

10. Find the side of a regular hexagon equal in area to the sum of three regular hexagons whose sides are 16, 30, and 32 respectively.

11. The sides of two squares are 64 and 120 respectively. Find the side of a square equivalent to their difference.

12. *Prove:* The square constructed on the diameter of a circle is equivalent to twice the area of the square inscribed in the circle.

13. An equilateral triangle and a right isosceles triangle are constructed on the hypotenuse of a right triangle. Find the ratio of their areas.

14. If b is the base and S one of the equal sides of an isosceles triangle, prove that the area of the triangle is equal to $\frac{1}{4}\, b\sqrt{4\,S^2 - b^2}$.

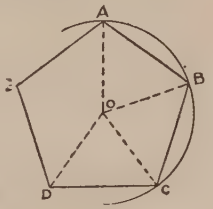
15. Given two similar triangles. Construct a triangle equivalent to the sum of the given triangles and similar to them.

16. Two similar triangles have corresponding sides of 8 in. and 15 in., respectively. How long is the corresponding side of a triangle which is similar to them and equivalent to their sum?

17. Show how to construct a square equivalent to three times a given square.

18. Find the diameter of a water-pipe whose cross-sectional area is equivalent to the sum of the cross-sectional areas of two other similar pipes whose diameters are 12 in. and 18 in. respectively.

(V). *A circle can be circumscribed about any regular polygon.*



Given regular polygon $ABCDE$.

To prove that a circle can be circumscribed about it.

Proof

1. A circle can be drawn through A, B, and C. Let O be the center of this circle. Draw OA, OB, OC, OD, etc.

2. $\angle ABC = \angle BCD$.

3. $\angle OBC = \angle OCB$.

4. $\therefore \angle ABO = \angle OCD$.

5. $OB = OC$.

6. $AB = CD$.

7. $\therefore \triangle ABO \cong \triangle OCD$.

8. $OD = OA$ (a radius).

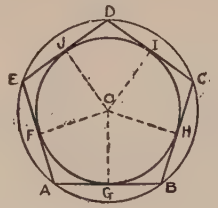
9. $\therefore$ the circle passes through D.

10. In the same way, we can prove that the circle passes through the remaining vertices of the polygon.

11. Hence a circle can be circumscribed about the regular polygon.

Cor. *Each central angle of a regular polygon of n sides is equal to $\frac{4}{n}$ right angles.*

(V). *A circle can be inscribed in any regular polygon.*



Given regular polygon $ABCDE$.

To prove that a circle can be inscribed in it.

Proof

1. Circumscribe a circle about the polygon and let its center be O.

2. Draw OF, OG, OH, etc., $\perp$ respectively to EA, AB, BC, etc.

3. OF, OG, OH, etc., are all equal.

4. Draw a circle with O as center and OF as radius.

5. This circle will pass through G, H, I, etc.

6. Then OG, OH, and OI will also be radii.

7. $\therefore$ AB, BC, CD, etc., will be tangent to this circle.

8. Hence the circle is inscribed in the polygon.

REGULAR POLYGONS

EXERCISE 45

1. How many degrees are there in the central angle of an equilateral triangle? square? regular pentagon? regular hexagon? regular octagon? regular decagon?

2. The apothem of a regular three-sided polygon is equal to one-half its radius.

3. The angle of any regular polygon is the supplement of the central angle of the same polygon.

4. The side of a regular hexagon is 12. Compute the length of its apothem.

5. A polygon is regular if its inscribed and circumscribed circles are concentric.

6. The radius of a regular inscribed polygon is a mean proportional between its apothem and the radius of the regular circumscribed polygon of the same number of sides.

7. Prove that the side of a regular hexagon is equal to the radius of the circumscribed circle.

8. Any radius of a regular polygon bisects the angle of the polygon.

9. A regular hexagon $ABCDEF$ is inscribed in a circle. BD and AE are drawn. Prove that $BDEA$ is a rectangle.

10. $ABCDE$ is a regular pentagon inscribed in a circle.
   (a) Prove that $\angle A$ is trisected by AC and AD.
   (b) At what angle do AD and CE intersect?
   (c) Prove that the tangent to the circle at A is parallel to CD.

11. Prove that the apothem of a regular polygon bisects the side to which it is drawn.

12. In regular polygon $ABCDEF$, prove that the diagonals AC, AD, and AE, divide $\angle BAF$ into four equal parts.

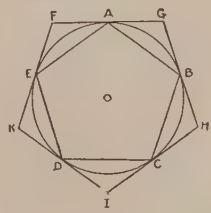
13. *Prove:* The diagonal AC of regular polygon $ABCDE$ is parallel to side DE.

14. Prove that if all the diagonals of a regular pentagon are drawn, another regular pentagon is formed.

15. *Prove:* An equiangular polygon circumscribed about a circle is regular.

16. If a base angle of an isosceles triangle is equal to three-fourths of the vertex angle, the base of the triangle is equal to the side of a regular pentagon inscribed in a circle whose radius is equal to the arm of the triangle.

(V). *If a circle is divided into any number of equal parts, the chords joining the successive points of division form a regular polygon inscribed in the circle; and the tangents at the points of division form a regular polygon circumscribed about the circle.*



Given circle ACE divided into equal arcs AB, BC, CD, etc., AB, BC, etc., are chords, and FAG, GBH, etc., are tangents.

To prove (*a*) $ABCDE$ is a regular polygon; (*b*) $FGHIK$ is a regular polygon.

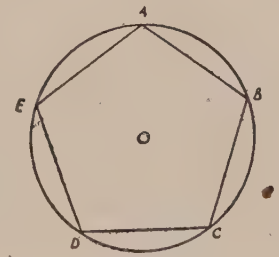
Proof (*a*)

1. $\overset{\frown}{AB}=\overset{\frown}{BC}=\overset{\frown}{CD}$, etc.
2. $\therefore \overset{\frown}{AC}=\overset{\frown}{BD}=\overset{\frown}{CE}$, etc.
3. $\angle ABC=\angle BCD=\angle CDE$, etc.
4. $\therefore$ $ABCDE$ is equiangular.
5. $AB=BC=CD$, etc.
6. $\therefore$ $ABCDE$ is equilateral.
7. $\therefore$ $ABCDE$ is a regular polygon.

Proof (*b*)

1. $\angle GAB=\angle GBA=\angle HBC=\angle HCB$, etc.
2. $AB=BC=CD$, etc.
3. $\therefore$ $\triangle AGB$, BHC, CID, etc., are all congruent.
4. $FA=GB=HC$, etc.
5. $AG=BH=CI$, etc.
6. $\therefore$ $FG=GH=HI$, etc.
7. $\therefore$ $FGHIK$ is equilateral.
8. $\angle G=\angle H=\angle I$, etc.
9. $\therefore$ $FGHIK$ is equiangular.
10. $\therefore$ $FGHIK$ is a regular polygon.

(V). *An equilateral polygon inscribed in a circle is regular.*



Given $\odot O$ and equilateral polygon $ABCDE$ inscribed in the circle.

To prove polygon $ABCDE$ is regular.

Proof

1. $AB=BC=CD$, etc.
2. $\overset{\frown}{AB}=\overset{\frown}{BC}=\overset{\frown}{CD}$, etc.
3. $\therefore$ $\overset{\frown}{ABC}=\overset{\frown}{BCD}=\overset{\frown}{CDE}$, etc.
4. $\therefore$ $\angle B=\angle C=\angle D$, etc.
5. $\therefore$ polygon $ABCDE$ is regular.

REGULAR POLYGONS—*Concluded*

EXERCISE 46

1. Show how to circumscribe a regular octagon about a circle; also a regular dodecagon.

2. Show how to inscribe in a circle a regular polygon similar to a given regular polygon.

3. The lines joining the midpoints of the radii of a regular pentagon form a regular pentagon whose area is one-fourth that of the first pentagon.

4. Prove that if each side of an equilateral triangle is trisected, and the points of division are joined in order, a regular hexagon is formed.

5. Construct eight points equidistant from each other and at a given distance from a given point.

6. The perimeters of regular polygons having the same number of sides are to each other as the diameters of the inscribed circles or as the diameters of the circumscribed circles.

7. If a regular polygon has an even number of sides, its opposite sides are parallel.

8. The diagonal AD of regular hexagon $ABCDEF$ is perpendicular to the diagonal CE and also bisects it.

9. The perimeter of a circumscribed equilateral triangle is twice that of an inscribed equilateral triangle.

10. The area of an inscribed regular hexagon is equal to twice the area of the inscribed equilateral triangle, and half the area of the circumscribed equilateral triangle.

11. *Prove:* If A, B, and C are three consecutive vertices of a regular polygon whose center is O, then OB is the perpendicular bisector of AC.

12. The diagonals joining the alternate vertices of a regular hexagon form by their intersections a regular hexagon whose area is one-third that of the original hexagon.

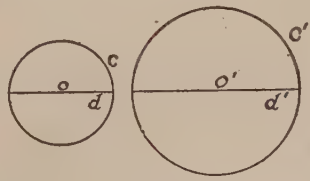
13. If $ABCDE$ is a regular polygon inscribed in circle O, prove that the diameter perpendicular to DE passes through B, and is also the perpendicular bisector of diagonal AC.

14. If the diagonals AC and BE of a regular inscribed pentagon $ABCDE$ intersect at F, prove that $\triangle ABF$ is isosceles.

15. In regular hexagon $ABCDEF$, diagonals CA and CE are drawn. Diagonal BD intersects CA and CE at H and K, respectively. Prove that $\triangle BHC$ and CKD are congruent.

16. In the figure of Ex. 15, prove that $\triangle BHC$ and CHK are equivalent.

(V). *The ratio of the circumference of a circle to its diameter is constant.*



Given any $\odot O$, with circumference C and diameter d.

To prove $\frac{C}{d}$ is constant, i.e., $\frac{C}{d}$ has the same value for all circles.

Proof

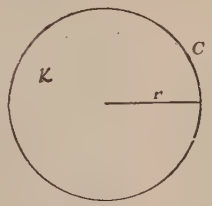
1. Construct a fixed circle O' with circumference C', and diameter d'.

2. $\frac{C}{C'}=\frac{d}{d'}$. **3.** or $\frac{C}{d}=\frac{C'}{d'}$.

4. But $\frac{C'}{d'}$ is constant.

5. $\therefore \frac{C}{d}$ is constant.

(V). *The circumference of a circle is equal to* $2\pi r$.



Given any circle. Let its circumference be C and its radius r.

To prove $C=2\pi r$

Proof

1. $\frac{C}{2r}$ is constant.

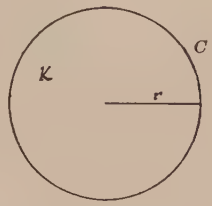
2. Let this constant number be represented by π (*pi*).

Then, $\frac{C}{2r}=\pi$.

3. $\therefore C=2\pi r$.

Cor. 1. *The length of an arc whose central angle is* a *degrees is* $\frac{a}{360}\times 2\pi r$.

(V). *The area of any circle is equal to* πr^2.



Given any circle. Let its circumference be C; its radius, r; and its area, K.

To prove $K=\pi r^2$.

Proof

1. $K=\frac{1}{2}C\times r$.

2. But $C=2\pi r$.

3. $\therefore K=\frac{1}{2}\times 2\pi r\times r$.

4. or $K=\pi r^2$.

Cor. 1. *The areas of two circles are to each other as the squares of their radii.*

Cor. 2. *The area of a sector whose central angle is* a *degrees is* $\frac{a}{360}\times\pi r^2$.

CIRCLE FORMULAS

Area of a circle	$K = \pi r^2 \ (\pi = \frac{22}{7})$
Circumference	$C = 2\pi r = \pi d$
Area of a sector	$= \frac{a^\circ}{360^\circ} \cdot \pi r^2$
Length of an arc of a°	$= \frac{a^\circ}{360^\circ} \cdot 2\pi r$
Area of a segment	= Sector $\pm$ triangle
Ratio of circumferences	$C : C' = r : r'$
Ratio of areas	$K : K' = r^2 : r'^2$

EXERCISE 47

1. Find the length in inches of an arc of 36° on a circle whose radius is (a) 3½ ft.; (b) 7 ft.

2. In a circle of radius 12, find the perimeter of a sector whose central angle is 47°.

3. Find the radius of a semicircle whose area is 88 sq. in.

4. The central angle of a sector is 80°, and its area is 18. Find the radius of the circle.

5. The radius of a circle is 10 ft., and the area of a sector of that circle is 12 π sq. ft. Find the number of degrees in the arc of the sector.

6. If the diameter of a given circle is 3 ft., find the diameter of a circle one-fourth as large.

7. The radius of a circle is 15 ft.; find (a) the radius of a circle whose circumference is twice that of the given circle; (b) the radius of a circle whose area is twice the area of the given circle.

8. Find (a) the area of the circle inscribed in an equilateral triangle whose side is 10 inches; (b) the area of the circle circumscribed about the same triangle.

9. If the radius of a certain circle is increased 1 ft., the area of the circle is doubled. Find the radius.

10. In a certain circle an arc of 22½° is $\frac{3\pi}{2}$ ft. long. Find the area of the circle.

11. A circular drive is 15 ft. wide, and the enclosed grass plot is 80 ft. in diameter. Find the area of the drive.

12. The minute hand and the hour hand of a watch are three-fourths of an inch and one-half of an inch long respectively. How many inches farther does the extremity of the minute hand travel in a day than the extremity of the hour hand?

XLIV

FORMULAS FOR AN EQUILATERAL TRIANGLE

Area $K = \frac{a^2}{4}\sqrt{3}$

Altitude $h = \frac{a}{2}\sqrt{3}$

Radius of circumscribed circle $= \frac{a}{3}\sqrt{3}$

Radius of inscribed circle $= \frac{a}{6}\sqrt{3}$

EXERCISE 48

1. The side of an equilateral triangle is 8. Find (a) the area; (b) the altitude; (c) the radius of the circumscribed circle; (d) the radius of the inscribed circle.

2. Find the area of a regular hexagon inscribed in a circle whose radius is 20.

3. Find the ratio of the areas of two equilateral triangles if the side of one is equal to the altitude of the other.

4. The altitude of an equilateral triangle is $5\sqrt{3}$. Find the perimeter and the area of the triangle.

5. The altitude of an equilateral triangle is 9 in. Find the area.

6. The area of an equilateral triangle is $48\sqrt{3}$. Find the altitude.

7. The area of an equilateral triangle is 16 sq. ft. Find the length of a side.

8. The side of an equilateral triangle is 10. Find (a) the radius of the circumscribed circle; (b) the radius of the inscribed circle.

9. Without using the above formulas, show that (a) the radius of the circle circumscribed about an equilateral triangle is double the radius of the circle inscribed in the triangle; (b) the radius of the inscribed circle is one-third the altitude of the triangle.

10. The radius of a circle is 6 in. Find the area of the segment of the circle subtended by a chord equal to the radius.

11. The radius of a circle is 2 in. Find the area of the segment cut off by a chord equal to a side of the inscribed equilateral triangle.

12. The altitude of an equilateral triangle is 19. Find the radii of the inscribed and the circumscribed circles.

13. Two equal circles, radius 12 in., intersect, and the common chord is equal to the radius. Find the area of the part common to the two circles.

XLV

FORMULAS FOR ANY TRIANGLE

Area $K = \frac{1}{2}b \cdot h = \sqrt{s(s-a)(s-b)(s-c)}$,
where $s = \frac{a+b+c}{2}$

Altitude to side a $h_a = \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}$

Median to side a $m_a = \frac{1}{2}\sqrt{2b^2+2c^2-a^2}$

Bisector of $\angle A$ $t_a = \frac{2}{b+c}\sqrt{bcs(s-a)}$

Radius of circumscribed circle $= \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$

Radius of inscribed circle $= \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}$

Projection of a **on** b $= \frac{a^2+b^2-c^2}{2b}$.

EXERCISE 49

1. Find the projection of the medium side upon the smallest side in a triangle whose sides are (a) 9, 12, 13; (b) 6, 8, 12.

2. The sides of a triangle are 28 in., 25 in., and 17 in. respectively. Find the area of the triangle and the length of the altitude to the longest side.

3. The sides of a triangle are 13, 14, and 15 respectively. Find the lengths of the three medians.

4. The sides of a triangle are 6, 9, and 12 respectively. Find the lengths of the three angle-bisectors.

5. The sides of a triangle are 9, 10, and 11 respectively. Find (a) the radius of the inscribed circle; (b) the radius of the circumscribed circle.

6. The area of a triangle is 84 sq. in., and one side is 14 in. Find the altitude to this side.

7. Find the area of a triangle if its perimeter is 21 in. and the radius of the inscribed circle is 1.6 in.

8. Find the area of a quadrilateral whose sides are 3 ft., 4 ft., 5 ft., and 6 ft., respectively, and one of whose diagonals is 7 ft.

9. Two sides of a parallelogram are 6 ft. and 9 ft., respectively. If one diagonal is 12 ft., find the other diagonal and the altitude of the parallelogram.

10. The radii of two intersecting circles are 15 and 20 respectively. The line of centers is 25. Find the length of the common chord.

11. Find the area and circumference of the circle circumscribed about a triangle whose sides are 10, 12, and 14 respectively.

12. An equilateral triangle whose side is 12 in. is inscribed in a circle. Find the area of the segment cut off by one side of the triangle.

XLVI

MISCELLANEOUS FORMULAS

Length of Lines	Areas
Right triangle, $a^2+b^2=c^2$.	Square, $K=a^2$.
Diagonal of square, $d=a\sqrt{2}$.	Rhombus, $K=\frac{1}{2}d\times d'$.
Side of square, $a=\sqrt{K}=R\sqrt{2}$.	Parallelogram, $K=b\times h$.
Side of equilateral triangle, $a=R\sqrt{3}$.	Trapezoid, $K=\frac{1}{2}h(b+b')$.
Side of regular hexagon, $a=R$.	Regular polygon, $K=\frac{1}{2}ap$.

In similar polygons $\begin{cases} p:p'=a:a'. \\ K:K'=p^2:p'^2=a^2:a'^2 \end{cases}$

Notation:

a, b, c = sides of triangle ABC.	p = perimeter.
d = diagonal.	b = base.
K = area.	h = altitude.
R = radius of circumscribed circle.	a = apothem of polygon, or side of square.

EXERCISE 50

1. Find the length of the diagonal of a rectangle whose sides are 221 ft. and 60 ft. respectively.

2. The hypotenuse of a right triangle is 12 in. and one arm is 8 in. Find the length of the other arm to the nearest hundredth.

3. A baseball diamond is a square 90 ft. on a side. Find to one decimal place the distance from the home plate to second base.

4. A tree is 2 ft. in diameter. Find to the nearest inch the side of the largest square column that can be cut from its trunk.

5. Find the side of an equilateral triangle inscribed in a circle whose radius is 10 in.

6. Corresponding sides of two similar polygons are 20 in. and 5 in. respectively. How many times is the perimeter of the first polygon as great as the perimeter of the second? How many times is the area of the first as great as the area of the second?

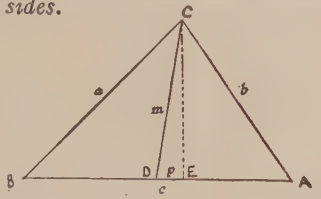
7. Two fields are similar in shape, and the area of the first is four times as great as that of the second. If fences of the same kind are to be built around them, how will the two costs compare?

8. The diagonals of a rhombus are 12 ft. and 16 ft. respectively. Find (a) the perimeter; (b) the area.

9. The altitude of a trapezoid is 6 ft. and one base is 15 ft. If the area is 97½ sq. ft., find the length of the other base.

10. A polygon is circumscribed about a circle whose radius is 8 ft. If the area of the polygon is 120 sq. ft., find its perimeter.

(III). *In any triangle, the square of one side plus four times the square of the corresponding median is equal to twice the sum of the squares of the other two sides.*



Given $\triangle ABC$ and m, the median to side c.

To prove $c^2+4m^2=2a^2+2b^2$.

Proof

1. Draw the altitude CE, and suppose E to fall between A and D. Let $DE=p$.

2. $\angle CDB$ is obtuse, and $\angle CDA$ is acute.

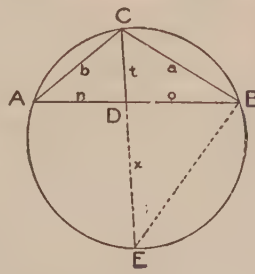
3. $\therefore a^2=\left(\frac{c}{2}\right)^2+m^2+2\left(\frac{c}{2}\right)p$.

4. $b^2=\left(\frac{c}{2}\right)^2+m^2-2\left(\frac{c}{2}\right)p$.

5. $a^2+b^2=2\left(\frac{c}{2}\right)^2+2m^2$.

6. *or* $2a^2+2b^2=c^2+4m^2$.

(III). *In any triangle, the product of two sides is equal to the square of the bisector of the included angle plus the product of the segments of the third side.*



Given $\triangle ABC$; t bisecting $\angle C$, and dividing AB into segments n and o.

To prove $ab=t^2+on$.

Proof

1. Circumscribe a circle about $\triangle ABC$. Produce CD to meet the circle in E. Draw BE, and let $DE=x$.

2. $\angle ACD=\angle ECB$.

3. $\angle A$ and $\angle E$ are each meas. by $\frac{1}{2}\overset{\frown}{CB}$.

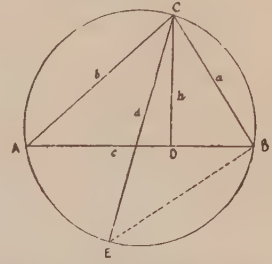
4. $\therefore \angle A=\angle E$.

5. $\triangle ACD\sim\triangle ECB$.

6. $\therefore b:t+x=t:a$. **7.** *or* $ab=t^2+tx$.

8. But $tx=on$. **9.** $\therefore ab=t^2+on$.

(III). *In any triangle, the product of two sides is equal to the altitude on the third side multiplied by the diameter of the circumscribed circle.*



Given d, a diameter of the circle circumscribed about $\triangle ABC$, and h, the altitude on side c.

To prove $ab=hd$.

Proof

1. Draw BE.

2. $\angle A=\angle E$.

3. $\angle CBE$ is a rt. $\angle$.

4. $\angle CBE=\angle CDA$.

5. $\triangle CBE\sim\triangle CDA$.

6. $\therefore a:h=d:b$.

7. $\therefore ab=hd$.

MISCELLANEOUS FORMULAS

(*Concluded*)

EXERCISE 51

1. The sum of the squares of the diagonals of any parallelogram is equal to the sum of the squares of the four sides.

2. From the relation $2a^2 + 2b^2 = c^2 + 4m^2$, derive a formula for m.

3. Two sides and a diagonal of a parallelogram are 7, 9, and 8 respectively. Find the length of the other diagonal.

4. The sides of a triangle are 10, 14, and 16 respectively. Find the lengths of the three medians.

5. The sides AB and AC of $\triangle ABC$ are 16 and 9 respectively, and the median drawn from C is 11; find side BC.

6. The sides of a triangle are 28, 42, and 50 respectively. Find the lengths of the segments into which side 50 is divided by the bisector of the opposite angle.

7. In the triangle of the preceding exercise, find the length of the bisector of the angle opposite side 50.

8. The sides of a triangle are 18, 9, and 21 respectively. Find the length of the bisector of the largest angle.

9. Two sides of a triangle are 5 and 12 respectively. The altitude on the third side is 4. Find the diameter of the circumscribed circle.

10. From a point A on a circle whose radius is 10, two chords are drawn, $AB = 8$ in. and $AC = 10$ in. Find the length of the perpendicular from A to BC.

11. In $\triangle ABC$, $AB = 10$, $BC = 7\frac{1}{2}$, and the projection of BC on AC is $4\frac{1}{2}$. Find the radius of the circumscribed circle.

12. From a point B on a circle, the perpendicular to a chord AC is 6 in. If AB is 10 in. and BC is 15 in., find the diameter of the circle.

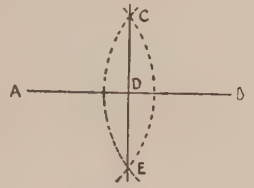
13. Find the length of the boundary of a circular segment between the side of a regular hexagon in a circle whose radius is 6 and the arc subtended by this side.

14. A circle whose diameter is 10 in. is inscribed in a quadrilateral whose perimeter is 60 in. Find the area of the quadrilateral.

15. The sides of a triangle are 12 in., 14 in., and 16 in. respectively. What is the area of the triangle formed by joining the midpoints of the sides of the triangle?

16. Two chords whose lengths are 24 in. and 10 in., respectively, are drawn from a point on a circle to the ends of a diameter. Find the length of the diameter and the distance of the point from the diameter.

(I). *To construct the perpendicular bisector of a line.*



Given line AB.

Required to construct the perpendicular bisector of AB.

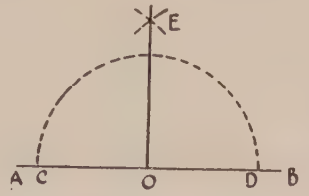
Construction

1. With A and B as centers, and equal radii greater than $\frac{1}{2}AB$, describe arcs intersecting at C and E.

2. Draw CE.

3. CE is the perpendicular bisector of AB.

(I). *At a given point in a given line, to erect a perpendicular to the line.*



Given point O in line AB.

Required to construct a perpendicular to AB at O.

Construction

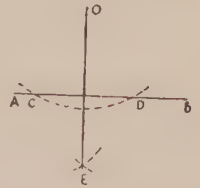
1. With O as center, and any radius, describe an arc intersecting AB in C and D.

2. With C and D as centers, and equal radii greater than OC, describe arcs intersecting at E.

3. Draw OE.

4. OE is the required perpendicular.

(I). *From a point outside a straight line to draw a perpendicular to the line.*



Given line AB and point O outside the line.

Required to construct a perpendicular to AB from O.

Construction

1. With O as center, and a radius sufficiently great, describe an arc cutting AB in C and D.

2. With C and D as centers, and equal radii greater than $\frac{1}{2}CD$, describe arcs intersecting at E.

3. Draw OE.

4. OE is the required perpendicular.

CONSTRUCTIONS

EXERCISE 52

1. Given an acute scalene triangle ABC. Construct (a) its three medians; (b) its three altitudes.

2. Given an obtuse triangle. Construct (a) its altitudes; (b) the perpendicular bisectors of its sides.

3. Construct the complement of a given acute angle.

4. Construct a square having its side equal to a given line.

5. Construct a right triangle, having given the arms.

6. Construct an isosceles right triangle, having given an arm.

7. Construct a square having its perimeter equal to a given line.

8. Construct a rectangle whose base is 3 in. and its altitude 2 in.

9. On a given line as diagonal, construct a square.

10. Construct a rhombus whose diagonals are equal, respectively, to two given unequal lines.

11. Construct a right triangle, given the hypotenuse and an arm.

12. Construct two unequal altitudes in a given parallelogram.

13. Construct a square, having given the sum of a diagonal and a side.

14. Draw any angle and construct its bisector. At the vertex of the angle construct a line perpendicular to the bisector. Prove that the angles which this perpendicular forms with the sides of the angle are equal.

15. Through a point within an angle, construct a line which shall form an isosceles triangle with the sides of the angle.

16. Construct an angle which shall be one and one-half times as large as a given angle.

17. Construct the supplement of one-half a given angle. How many degrees would this angle contain if the given angle is 65° 18′?

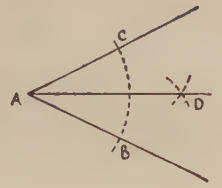
18. Construct the perpendicular bisector of a line which is parallel to the edge of your paper and about one-quarter inch distant from the edge.

19. Construct two lines which bisect each other at right angles. If the ends of these lines are joined in order, how can the area of the quadrilateral formed be found?

20. A baseball diamond is a square 90 ft. on a side. Construct a plan of such a figure, using a scale of $\frac{1}{16}$ in. to a foot. How far is it from first base to third base? Verify your answer by measurement.

21. A horse is tied by a rope 25 ft. long to one corner of a rectangular building 25 ft. long and 20 ft. wide. Using a scale of $\frac{1}{8}$ in. to 1 ft., make a drawing of the area over which the horse can graze.

(I). *To bisect a given angle.*



Given $\angle A$.

Required to bisect $\angle A$.

Construction

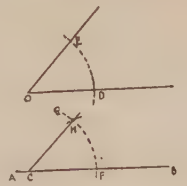
1. With A as center, and any radius, describe arcs cutting the sides of the angle at B and C.

2. With B and C as centers, and equal radii greater than $\frac{1}{2}BC$, describe arcs intersecting at D.

3. Draw AD.

4. AD bisects $\angle A$.

(I). *Through a given point on a given straight line, to draw a line making a given angle with the given line.*



Given point C in line AB, and $\angle O$.

Required to construct an angle at $C = \angle O$.

Construction

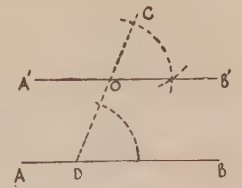
1. With O as center, and any radius, describe an arc cutting the sides of $\angle O$ at D and E.

2. With C as center, and the same radius, describe arc FG, cutting CB in F.

3. With F as center, and a radius equal to DE, describe an arc cutting FG at H.

4. Draw CH. **5.** $\angle HCF = \angle O$.

(I). *Through a given point to draw a parallel to a given line.*



Given line AB and point O.

Required to construct a parallel to AB through O.

Construction

1. Draw a line, COD, through O, meeting AB at D.

2. At O, construct $\angle COB' = \angle ODB$.

3. $A'B' \parallel AB$.

CONSTRUCTIONS—*Continued*

EXERCISE 53

1. Construct a parallelogram two of whose sides are 3 in. and 4 in. respectively, and having an angle of 45°. Construct also its altitudes.

2. In quadrilateral $ABCD$, $AB = AD = 2$ in., $\angle A = 90°$, $\angle B = 120°$, and $\angle D = 60°$. Construct the figure, showing all construction lines.

3. Construct an angle of $22\frac{1}{2}°$; $67\frac{1}{2}°$; $165°$.

4. Given two angles A and B. Construct an angle equal to (a) their sum; (b) their difference.

5. Construct the bisectors of two supplementary-adjacent angles. What angle do the bisectors make with each other?

6. Show how to trisect a right angle.

7. Construct a parallelogram, having given two adjacent sides and the included angle.

8. Through a given point P, draw a line which shall make an angle of 45° with a given line AB, (a) when P is on AB; (b) when P is outside AB.

9. Through each vertex of a triangle, construct a line parallel to the opposite side.

10. Construct one-half the supplement of a given angle. How many degrees would this angle contain if the given angle is 65° 18′?

11. Construct a triangle, having given (a) the three sides; (b) two sides and the included angle; (c) two angles and the included side.

12. Construct an isosceles triangle, having given (a) the base and a base angle; (b) an arm and a base angle.

13. From a given point outside a line draw a line which shall make a given angle with the given line.

14. Construct a trapezoid, given the bases, one of the non-parallel sides, and the angle included between this side and the base.

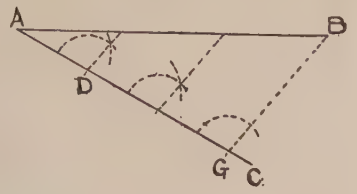
15. Construct an isosceles trapezoid having its upper base $1\frac{1}{2}$ in., its non-parallel side 2 in., and its lower base angle 30°. How long is the altitude of the trapezoid?

16. Construct two parallel lines and the bisectors of a pair of corresponding angles. Are these bisectors parallel? Why?

17. Through a given point within a circle, construct a chord equal to a given chord.

18. In $\triangle ABC$, construct a line DE parallel to BC so that $DE = EC$.

(I). *To divide a line into any number of equal parts.*



Given line AB.

Required to divide AB into n equal parts.

Construction

1. From A draw any line AC.

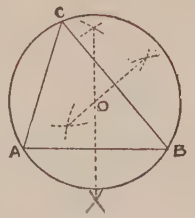
2. On AC lay off any segment, as AD, n times in succession.

3. Connect the last point of division, as G, with B.

4. Through the other points of division on AC, draw parallels to BG.

5. These parallels divide AB into n equal parts.

(II). *To circumscribe a circle about a given triangle.*



Given $\triangle ABC$.

Required to circumscribe a circle about $\triangle ABC$.

Construction

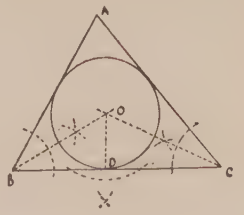
1. Draw the $\perp$ bisectors of two sides of the $\triangle$, as AB and BC.

2. These $\perp$ bisectors will intersect at some point, as O.

3. With O as center, and OA as radius, describe circle ACB.

4. This circle is the required circle.

(II). *To inscribe a circle in a given triangle.*



Given $\triangle ABC$.

Required to inscribe a circle in $\triangle ABC$.

Construction

1. Bisect two angles, as $\angle B$ and $\angle C$.

2. These bisectors will meet at some point O.

3. From O draw $OD \perp BC$.

4. With O as center, and OD as radius, draw a circle.

5. This circle is the required circle.

CONSTRUCTIONS—*Continued*

EXERCISE 54

1. Construct an equilateral triangle, having given the perimeter.

2. Using as a chord a line 1½ in. long, construct a segment of a circle in which an angle of 45° can be inscribed.

3. Construct a line two-thirds as long as a given line.

4. Find by construction the center of a given circle.

5. Circumscribe a circle about a given right triangle.

6. Construct a circle tangent to two sides of a triangle prolonged, and also tangent externally to the third side.

7. Inscribe a circle in a given sector of a circle.

8. Prove that a circle can be inscribed in a rhombus.

9. Circumscribe an isosceles triangle about a given circle, having given as base a line greater than the diameter of the circle.

10. Through a given point within a circle draw a chord whose length shall be equal to that of a given line which is less than the diameter.

11. Construct a circle having a given radius which shall be tangent to a given circle at a given point, (a) externally; (b) internally.

12. Circumscribe a right triangle about a given circle, having given one arm greater than the diameter of the circle.

13. On a given line as chord, construct a segment of a circle which shall contain an angle of 105°.

14. Construct within a given triangle a circle which shall be tangent to two sides of the triangle and also tangent to the inscribed circle of the triangle.

15. Construct two tangents to a circle which shall make with each other an angle of 45°.

16. Construct a tangent to a given circle, (a) parallel to a given line; (b) perpendicular to a given line.

17. Can a circle be circumscribed about any given rectangle? Give the construction.

18. Construct a tangent to a circle which shall make an angle of 60° with a given line.

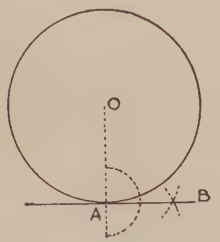
19. Construct a circle tangent to a given line, and having its center at a given point not on the line.

20. Construct a line whose distance from a given point P shall be equal to a given distance, d.

21. Find by construction a point in one side of a quadrilateral which is equidistant from the ends of the opposite side.

22. Through a given point without an angle, construct a line forming an isosceles triangle with the sides of the angle.

(II). *To draw a tangent at a given point on a given circle.*



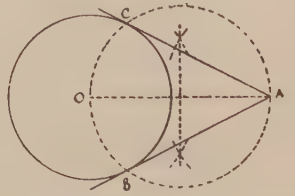
Given $\odot O$ and point A on the circle.

Required to draw a tangent at A.

Construction

1. Draw radius OA.
2. Draw $AB \perp OA$ at A.
3. AB is the required tangent.

(II). *To draw a tangent to a circle from a given point outside the circle.*



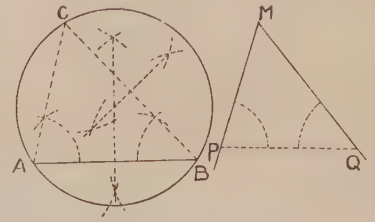
Given $\odot O$ and point A outside.

Required to draw a tangent to the circle from A.

Construction

1. Connect A and O.
2. Draw a circle on OA as diameter, cutting the given circle at C and B.
3. Draw AB and AC.
4. AB and AC are both tangent to $\odot O$.

(II). *On a given straight line as chord, to construct a segment of a circle, such that any angle inscribed in it shall be equal to a given angle.*



Given line AB and $\angle M$.

Required to construct a segment of a circle on AB as chord which shall contain any angle equal to $\angle M$.

Construction

1. Take any two points P and Q in the sides of $\angle M$.
2. Draw PQ.
3. On AB as base, construct $\triangle ABC$ so that $\angle A = \angle P$, and $\angle B = \angle Q$.
4. Circumscribe a circle about $\triangle ABC$.
5. Segment ACB is the required segment.

CONSTRUCTIONS—*Continued*

Construction of Figures from Given Parts

Method

1. Draw a free-hand figure and mark all known lines and angles.
2. Try to discover in the figure a triangle in which three parts (at least one of which is a side) are known. Construct this triangle first.

Note.—Avoid special figures, such as isosceles or equilateral triangles, parallelograms, etc., unless the problem specifically mentions such a figure.

EXERCISE 55

1. Construct a triangle, given a side, the median to that side, and the altitude to another side. (*Hint.*—Draw a model triangle ABC with altitude CE and median AD. Start with $\triangle CEB$, then locate point D, next point A, etc.)

2. Construct an isosceles triangle, given the base and the radius of the inscribed circle.

3. Construct a triangle, two of whose sides are 4 in. and 3 in. respectively, and the altitude to the third side is 2 in. (*Two solutions.*)

4. Construct a triangle whose base is $3\frac{1}{2}$ in. long, one base angle 45°, and the altitude to the base $1\frac{1}{2}$ in. long.

5. Construct a triangle, given the middle points of the sides.

6. Construct a parallelogram, given a side and the two diagonals.

7. Construct an isosceles triangle, given the base and the vertex angle.

8. Construct a right triangle, given an arm and the altitude on the hypotenuse.

9. Construct a rectangle, given one side and the diagonal.

10. Construct a right triangle, given the segments of the hypotenuse made by the bisector of the right angle.

11. Construct a triangle, given two sides and the median to one of these sides.

12. Given one side of a triangle, the altitude to that side, and the altitude to another side. Construct the triangle.

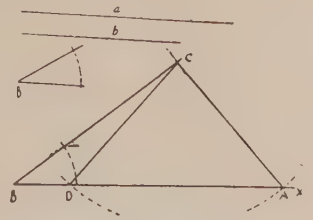
13. Construct an equilateral triangle whose altitude is equal to a given line h.

14. Construct a trapezoid, having given the four sides.

15. Construct a triangle, having given the base, a base angle, and the angle which the median to the base makes with the base.

16. Construct a triangle, having given the base, the vertex angle, and the median to the base.

(II). *To construct a triangle, having given two sides and the angle opposite one of them.*



Given lines a and b and $\angle B$.

Required to construct a triangle having an angle equal to $\angle B$, the opposite side equal to b, and another side equal to a.

Construction

1. Draw an indefinite line BX.

2. At B construct an angle equal to the given angle.

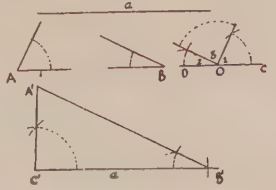
3. Lay off $BC = a$.

4. With C as center, and b as radius, describe an arc intersecting BX at D and A.

5. Draw CD and CA.

6. $\triangle BCD$ and BCA both contain the given parts.

(II). *To construct a triangle, having given one side, one adjacent angle, and the opposite angle.*



Given side a, $\angle A$, and $\angle B$.

Required to construct a triangle, having a side equal to a, an adjacent angle $= \angle B$, and the opposite angle $= \angle A$.

Construction

1. Draw a st. $\angle$, as $\angle DOC$.

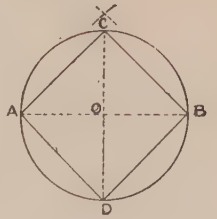
2. At O, construct $\angle 1 = \angle A$, and $\angle 2 = \angle B$.

3. $\angle 3$ will be the third angle of the required triangle.

4. Construct $\triangle A'B'C'$, having $C'B' = a$, $\angle C' = \angle 3$, and $\angle B' = \angle B$.

5. $A'B'C'$ is the required triangle.

(V). *To inscribe a square in a circle.*



Given circle O.

Required to inscribe a square in $\odot O$.

Construction

1. Draw diameter AB.

2. Draw diameter $CD \perp AB$.

3. Connect, in order, A, C, B, and D.

4. $ACBD$ is the required square.

CONSTRUCTIONS—*Continued*

EXERCISE 56

1. Inscribe a regular octagon in a given circle.

2. Circumscribe a square about a given circle.

3. Construct a regular octagon, having given a side.

4. If r is the radius of a circle and s the side of the inscribed square, prove that $r = \frac{s}{2}\sqrt{2}$.

5. Show how to construct a regular octagon from a given square by drawing lines across the corners.

6. Construct a right triangle, given the hypotenuse and one arm.

7. Construct an isosceles triangle, given the base and the altitude on an arm.

8. Construct an isosceles triangle, given the base and the vertex angle.

9. Construct a right triangle, given an arm and the opposite angle.

10. Construct a triangle in which two of the sides are 1 in. and 1½ in. respectively, and the angle opposite the latter side is 45°.

11. Construct a triangle in which two of the sides are 1½ in. and ¾ in. respectively, and the angle opposite the latter side is 30°.

12. Construct a triangle having a side 2 in. long, an adjacent angle equal to 45° and the opposite angle equal to 67½°.

13. Inscribe a square in a circle of radius 2 in. Find the length of one of its sides and also its area.

14. Construct an isosceles triangle, having given the base angle and the altitude to an arm.

15. Construct a rhombus, having given its base and altitude.

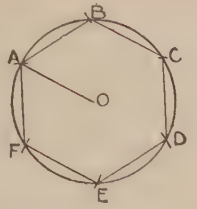
16. Within a given right triangle inscribe a square, one angle of which coincides with the right angle of the triangle, the vertex of the opposite angle lying on the hypotenuse.

17. Construct triangle ABC, having given $\angle A$, side b, and the altitude to side a.

18. Construct triangle ABC, having given side a, the altitude to a, and the median to a.

19. Construct triangle ABC, having given $\angle C$, side b, and the bisector of $\angle C$.

(V). *To inscribe a regular hexagon in a circle.*



Given circle O.

Required to inscribe a regular hexagon in the circle.

Construction

1. Draw a radius OA.

2. With A as center, and OA as radius, draw an arc cutting the circle at B.

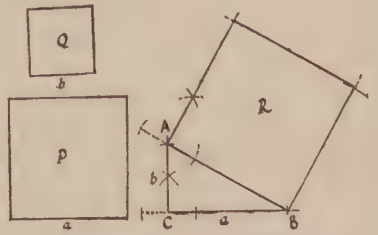
3. With B as center, and the same radius, draw an arc cutting the circle at C.

4. Repeat this process until point F is obtained.

5. Connect points A–F in order.

6. Then $ABCDEF$ is the required regular hexagon.

(IV). *To construct a square equivalent to the sum of two given squares.*



Given squares P and Q, having sides a and b respectively.

Required to construct a square equivalent to the sum of P and Q.

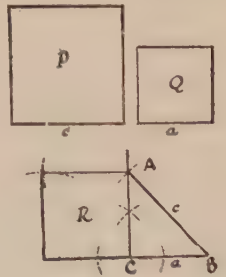
Construction

1. Construct a right triangle ABC having arm AC equal to b and arm BC equal to a.

2. Construct square R having hypotenuse AB as a side.

3. R is the required square.

(IV). *To construct a square equivalent to the difference of two given squares.*



Given squares P and Q, having sides c and a respectively.

Required to construct a square equivalent to the difference of P and Q

Construction

1. Construct a right triangle ABC having its hypotenuse $AB=c$, and an arm $CB=a$.

2. Construct square R on the other arm AC.

3. R is the required square.

CONSTRUCTIONS—*Continued*

EXERCISE 57

1. Circumscribe a regular octagon about a given circle.

2. Construct a square equal to the sum of three given squares.

3. Inscribe an equilateral triangle in a given circle.

4. Circumscribe an equilateral triangle about a given circle.

5. If R is the radius of a circle, prove that the side of the inscribed equilateral triangle is equal to $R\sqrt{3}$.

6. Construct a regular hexagon having each side equal to 1½ in.

7. Construct a square equivalent to the difference between a given square and the sum of two other given squares.

8. Inscribe a regular polygon of 12 sides in a circle whose radius is 1½ in.

9. Construct a square having twice the area of a given square.

10. Construct a square having one-half the area of a given square.

11. Construct a square equivalent to the sum of three squares whose sides are 1, 2, and 2½ in. respectively. Find the number of inches in one side of this square.

12. If a and b are given lines, construct a line x, so that $x = \sqrt{a^2 + b^2}$.

13. If a and b are given lines, construct a line x, so that $x = \sqrt{a^2 - b^2}$.

14. Circumscribe a regular hexagon about a given circle.

15. Construct a line whose length is (a) $\sqrt{2}$ in.; (b) $\sqrt{3}$ in.

16. If chord AB in a given circle is equal to the radius, and chord AC is the side of a regular decagon, prove that chord CB will be a side of a regular polygon of 15 sides.

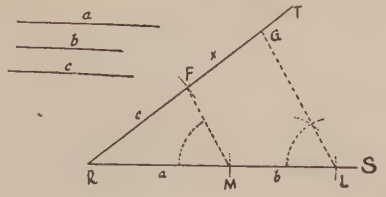
17. If a square is inscribed in a circle whose radius is R, the side of the square is equal to $R\sqrt{2}$.

18. Construct a regular hexagon by trisecting the sides of an equilateral triangle and connecting the trisection points in order. Give proof.

19. Inscribe a square in a given sector whose central angle is 90°.

20. An iron plate 1 ft. in diameter is to have 12 holes bored in it, equidistant from each other and 1 in. from the boundary of the plate. Make a scale drawing ¼ in. to the ft., showing the location of the centers of the holes.

(III). *To construct the fourth proportional to three given lines.*



Given lines a, b, and c.

Required to construct the fourth proportional to a, b, and c.

Construction

1. From any point as R, draw two indefinite lines, RS and RT.

2. On RS lay off $RM=a$ and $ML=b$.

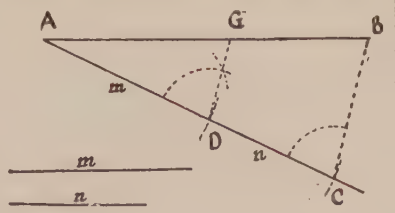
3. On RT lay off $RF=c$.

4. Connect F and M.

5. Draw $LG \parallel FM$.

6. FG is the required fourth proportional.

(III). *To divide a line into segments proportional to two given lines.*



Given lines AB, m, and n.

Required to divide AB into segments proportional to m and n.

Construction

1. Draw AC making any angle with AB.

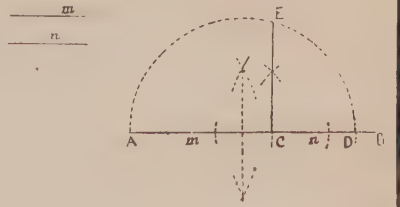
2. On AC lay off $AD=m$, and $DC=n$.

3. Connect B and C.

4. Through D draw a line $\parallel BC$, meeting AB in G.

5. Point G divides AB as required.

(III). *To construct the mean proportional between two given lines.*



Given lines m and n.

Required to construct the mean proportional between m and n.

Construction

1. Draw an indefinite line AB.

2. On AB lay off $AC=m$, and $CD=n$.

3. Construct a semicircle having AD as its diameter.

4. Draw a perpendicular to AD at C, meeting the circle at E.

5. CE is the required mean proportional.

CONSTRUCTIONS—*Continued*

EXERCISE 58

1. Given three lines, a, b, and c. Construct a line x, so that $x:b=c:x$.

2. Construct a line equal to $\frac{ab}{c}$, if a, b, and c are given lines.

3. Construct a line equal to $\frac{a^2}{c}$, if a and c are given lines.

4. Given lines a and b. Construct a line equal to $\sqrt{ab}$.

5. Divide a given line into two parts so that one part shall be $2\frac{1}{2}$ times as long as the other. How long will each part be if the entire line is 9 in. long?

6. If a and b are two given lines, construct a line x so that $x:b=b:x$.

7. Divide a given line into three segments which are to each other as $2:3:4$.

8. If a, b, and c are three given lines, construct a line x so that $a:b=x:c$.

9. Divide a line 3 in. long into two segments which shall have the same ratio as the side of a square has to its diagonal.

10. Given three lines, a, b, and c. Construct two lines whose sum shall be equal to a, and which shall be to each other as $b:c$.

11. If a and b are two given lines, construct a line x which is equal to $\sqrt{6ab}$.

12. Construct a line equal to $a\sqrt{2}$, if a is a given line.

13. Construct the mean proportional between two lines $\frac{3}{4}$ in. and $1\frac{1}{4}$ in. long respectively. Find the numerical value of your answer.

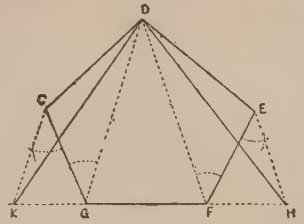
14. If a is a given line, construct a line x, so that $x=\sqrt{3\,a^2}$.

15. If a is a given line, construct a line x, so that $x=\frac{a}{2}\sqrt{3}$.

16. (a) Construct the fourth proportional to three lines whose lengths are $1\frac{1}{4}$ in., 2 in., and $2\frac{1}{4}$ in., respectively. (b) Calculate, arithmetically, the length of this fourth proportional and compare this with the result obtained by measurement.

17. Given three lines a, b, and c. Construct two lines whose difference shall be equal to a, and which shall be to each other as $b:c$.

(IV). *To transform a polygon into a triangle.*



Given polygon $CDEFG$.

Required to construct a triangle equivalent to $CDEFG$.

Construction

1. Connect two alternate vertices, as D and G.

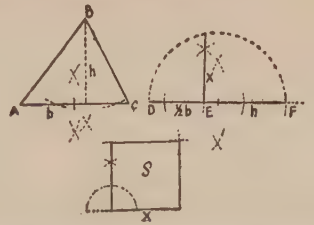
2. Through C draw a parallel to DG, meeting FG produced in K.

3. Figure $KDEF$ is equivalent to $CDEFG$ and has one less side.

4. Repeat this process until $\triangle KDH$ is obtained.

5. $\triangle KDH$ is the required triangle.

(IV). *To construct a square equivalent to a given triangle.*



Given $\triangle ABC$ with base b and altitude h.

Required to construct a square equivalent to $\triangle ABC$.

Construction

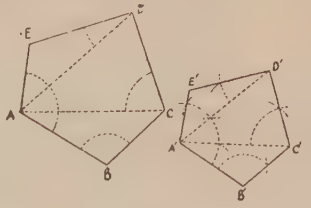
1. Bisect base b.

2. Construct the mean proportional between $\frac{1}{2}b$ and h, as x.

3. Construct square S, having x as side.

4. S is the required square.

(III). *To construct a polygon similar to a given polygon upon a line corresponding to a side of the given polygon.*



Given polygon $ABCDE$ and line $A'B'$ corresponding to side AB.

Required to construct a polygon on $A'B'$ similar to $ABCDE$.

Construction

1. Draw diagonals AC and AD.

2. Construct $\triangle A'B'C'$ with $\angle B' = \angle B$, and $\angle B'A'C' = \angle BAC$.

3. Construct $\triangle A'C'D'$ with $\angle A'C'D' = \angle ACD$, and $\angle C'A'D' = \angle CAD$.

4. Construct $\triangle A'D'E'$ with $\angle A'D'E' = \angle ADE$, and $\angle D'A'E' = \angle DAE$.

5. $A'B'C'D'E'$ is the required polygon.

CONSTRUCTIONS—*Concluded*

EXERCISE 59

1. Construct a square equivalent to a given rectangle.

2. Construct a square equivalent to a given parallelogram.

3. Construct a square equivalent to a given quadrilateral. (*Hint.*—First transform the quadrilateral into a triangle.)

4. Given a pentagon with one side equal to 1½ in. Construct a similar pentagon having the side corresponding to the 1½ in. side of the first equal to ½ in. What is the ratio of the areas of the two polygons?

5. Construct a triangle equivalent to a given trapezoid.

6. Construct a square whose area shall be three times as great as that of a given square.

Analysis.—Let x be the side of the required square and a the side of the given square. Then—

$$x^2 = 3a^2 \text{ or } 3a : x = x : a$$

7. Construct a square whose area shall be one-half that of a given square.

8. On a given line as base, construct a rectangle equivalent to a given rhombus.

Analysis.—Let m be the given line, b the base, and h the altitude of the rhombus, and x the unknown altitude of the rectangle. Then $bm = hx$, or $h : b = m : x$

9. On a given line as base construct a rectangle equivalent to a given triangle.

10. Construct a triangle similar to a given triangle and having twice the area of the given triangle.

Analysis.—Let a be a side of the given triangle and x the corresponding side of the required triangle. Also let S be the area of the given triangle, and $2S$ the area of the required triangle. Then $S : 2S = a^2 : x^2$; $x^2 = 2a^2$; or $2a : x = x : a$

11. Divide a triangle into two equivalent parts by a line parallel to one of its sides.

12. Construct a triangle equivalent to a given triangle and having a given line as base. Is more than one such triangle possible?

13. Construct an isosceles triangle having a given base and equivalent to a given square.

14. Construct a square whose area shall be equal to two-thirds the area of a given rectangle.

15. Construct a rectangle having a given altitude and whose area is equal to that of a given square. Consider the special case when the side of the square is 4 and the given altitude is 2.

COMPLETION TEST

EXERCISE 60

1. The mean proportional between 8 and 18 is ....

2. The apothem of a square inscribed in a circle of radius 4 in. is .... in.

3. A base angle of an isosceles triangle is 40°. The angle at the vertex is .... degrees.

4. In a circle of radius 9 in., a chord subtending an arc of 120° is .... in. from the center.

5. At five o'clock the hands of a watch form an angle of .... degrees.

6. Two squares are similar because their angles are .... and their corresponding sides are ....

7. A triangle whose base is 3 times its altitude has an area of 24 sq. yd. The altitude of the triangle is .... yd.

8. Two chords intersecting in a circle cut off four consecutive arcs having the continued ratio of 3 : 4 : 5 : 6. The acute angle formed by the chords has .... degrees.

9. A tangent drawn at a vertex of an inscribed regular hexagon makes an angle of .... degrees with one side.

10. The formula for the area of a circle in terms of its radius is ....

11. If the perimeter of an equilateral triangle is 12, its altitude is ....

12. If the area of a circle is 36 π sq. in., then its circumference is .... inches.

13. The perpendicular bisectors of the three sides of a triangle meet in a point that is equidistant from the .... of the triangle.

14. The two diagonals of a parallelogram divide the figure into four .... triangles.

15. The areas of two similar triangles are to each other as the .... of corresponding altitudes.

16. An inscribed angle is measured by .... its intercepted arc.

17. If the area of an inscribed square is 36 sq. ft., then the area of the circumscribed square is .... sq. ft.

18. The formula connecting the diagonal of a square with one side, s, is $d =$ ....

19. The line of centers of two circles tangent to each other passes through the ....

20. *Complete*: If from a point without a circle, a tangent and a secant are drawn, ...

COMPLETION TEST—*Continued*

EXERCISE 61

1. The diagonals of a rhombus are .... to each other.

2. If the diagonals of a parallelogram are equal, the figure is a ....

3. The longer base of a trapezoid is 20 in., and each of the other three sides is 10 in. Each lower base angle has .... degrees.

4. The fourth proportional to 9, 12, and 21 is ....

5. The vertex angle of an isosceles triangle is 100°. The angle formed by the bisectors of the base angles is .... degrees.

6. The bisectors of the angles of a triangle meet in a point that is equidistant from the .... of the triangle.

7. If two chords intersect within a circle the product of .... is equal to ....

8. The locus of the centers of all circles touching two parallel lines is....

9. Two lines perpendicular to a third line are .... to each other.

10. The number of acute angles in a triangle must be at least ....

11. The locus of all points 2 in. from a given line is ....

12. Two angles whose sum is two right angles are said to be .... angles.

13. The distance from a point to a circle is measured on the line that is ....

14. Each exterior angle of a regular decagon has .... degrees.

15. The length of an arc of 80° in a circle whose radius is 3 in. is .... π in.

16. In $\triangle ABC$, $\angle A = 50°$ and $\angle B = 60°$. The longest side is ....

17. A pentagon has .... diagonals.

18. In a circle whose radius is 10 in., a 12 in. chord is .... in. from the center.

19. The difference between two sides of a triangle is .... than the third side.

20. The perimeter of a square equal in area to a rectangle 16 in. $\times$ 9 in. is .... in.

COMPLETION TEST—*Continued*

EXERCISE 62

1. A square whose diagonal is 12 yd. has an area of .... sq. yd.

2. Triangles having equal bases are to each other as their ....

3. If $ab = cd$, a proportion can be written as follows, a: .... = c: ....

4. The locus of the midpoints of all equal chords in a circle is ....

5. If two perpendicular chords intersect inside a circle, the sum of a pair of opposite arcs is equal to ....

6. The ratio, to the nearest tenth, of a side of a square to its diagonal is ....

7. In right $\triangle ABC$, right angle at C, AC is one-half of the hypotenuse. $\angle A =$ .... degrees.

8. The converse of the theorem, "The base angles of an isosceles triangle are equal," is ....

9. In order to prove that the sum of the angles of a triangle is 180°, we draw a line through .... and .... a side.

10. An angle that is greater than its supplement is .... angle.

11. The difference between the supplement and the complement of an angle is .... degrees.

12. If two chords are unequal, the greater chord is at the .... distance from the center.

13. In constructing the bisector of an angle, we use .... different points as centers of circles.

14. If the area of one circle is 16 times the area of a second circle, then the radius of the first circle is .... times that of the second circle.

15. Two circles can not intersect in more than .... points.

16. The altitude upon the hypotenuse of a right triangle divides it into two .... triangles.

17. One-half of an obtuse angle is .... angle.

18. If the vertex angle of an isosceles triangle is two-thirds of a right angle the triangle is ....

19. If the median of a triangle is one-half the side to which it is drawn, the triangle is .... triangle.

20. The locus of the centers of all circles passing through two given points A and B is the .... of the ....

COMPLETION TEST—*Concluded*

EXERCISE 63

(Regents)

In each of the following write the word or words which complete the statements.

1. If sides AB and CD of quadrilateral $ABCD$ are equal and parallel then $ABCD$ must be a ....

2. When two lines are cut by a transversal, if a pair of .... angles are equal, the two lines are parallel.

3. The sides of a right triangle are 3, 4 and 5. Its area is ....

4. The hypotenuse of a right triangle whose sides are 5 and 12 is ....

5. If a is the apothem and s the side of a regular hexagon its area is ....

6. In similar triangles ABC and $A'B'C'$, $AB : A'B' = 3 : 2$, then triangle ABC : triangle $A'B'C'$ = ... : ....

7. If the parallel sides of a trapezoid are 12 and 16, and the altitude is 4.5, the area of the trapezoid is ....

8. If D and E are points in the sides AC and BC of triangle ABC such that $CD : DA = CE : EB$, then triangle CDE is .... to triangle ABC.

9. CD is the altitude on the hypotenuse of the right triangle ABC. If $AD = 30$ and $BD = 3\ 1/3$, then CD = ...

10. Chords AB and CD of a circle intersect at P. If $AP = 6$, $CP = 8$, $BP = 16\ 2/3$, then DP = ...

11. ABC is a triangle inscribed in a circle. If angle $A = 40°$, angle $B = 65°$, then the arc AB contains .... degrees.

12. Points A, B, C, D and E divide a circle into equal arcs. The chords joining successive points of division form a ....

13. An angle formed by a tangent and a chord is measured by ....

14. A circle can be circumscribed about a polygon if the polygon is ....

15. In triangle ABC, if AB is greater than BC is greater than AC the smallest angle is angle ....

16. In triangle ABC, if angle A is less than angle B is less than angle C, the longest side is side ....

17. In triangle ABC, angle $A = 2$ angle B, and angle $B = 2$ angle C; then angle A contains .... degrees.

18. If two isosceles triangles have a common base, the line determined by their vertices is ....

19. The center of the circle inscribed in a triangle is equidistant from the three ....

20. The diameter DC of a circle is perpendicular to chord AB. If arc AC contains 70°, then arc BD contains .... degrees.

21. The distances of two equal chords from the center of a circle are ....

22. A line is parallel to one side of a triangle if it divides the other two sides ....

23. Two triangles in which the ratios of corresponding sides are equal are ....

24. The area of a circle whose radius is 2a is ....

25. The center of the circle circumscribed about a given triangle is the point of intersection of the ....

26. If the vertex angle of an isosceles triangle contains 2a degrees, each of the other angles contains .... degrees.

27. Pi is the ratio of the .... to the .... of the circle.

28. Each side of a square inscribed in a circle is equal to .... times the radius of the circle.

29. The locus of points equidistant from the sides of a given angle is ....

30. The opposite angles of an inscribed quadrilateral are ....

In exercises 31–33 *underscore the correct word.*

31. "Through a given point one and only one line can be drawn parallel to a given line" is a (definition, postulate, theorem).

32. "If two angles of a triangle are equal, the sides opposite these angles are equal" is a (definition, postulate, theorem).

33. "The point of intersection of the bisectors of the angles of a triangle is the *incenter* of the triangle" is a (definition, postulate, theorem).

TRUE-FALSE TEST

EXERCISE 64

State whether each of the following statements is true or false. Give reasons.

1. Doubling the length and the width of a rectangle doubles its area.

2. The sum of the diagonals of a parallelogram is less than the sum of the four sides.

3. The altitude upon the hypotenuse of a right triangle is less than the hypotenuse.

4. The following lengths may be used to form a triangle: 7 in., 2½ in., and 4.5 in.

5. Two triangles whose sides are respectively 8, 6, and 4 and 2, 3, and 4 are similar.

6. A square can be constructed equal to any polygon.

7. Doubling the radius of a circle doubles its circumference.

8. The locus of all points 1 inch from a circle of radius 6 inches is a circle concentric with the given circle and having a 7-inch radius.

9. If in $\triangle ACE$ a line BD is drawn cutting AC in B and AE in D so that $AB : BC = 3 : 2$, and $AD = 6$ and $AE = 10$, then $BD \parallel CE$.

10. If the number of sides of a polygon is tripled the sum of its interior angles is tripled.

11. The perimeter of an inscribed square is greater than the perimeter of an equilateral triangle inscribed in the same circle.

12. The diagonals of an isosceles trapezoid bisect each other.

13. Regular polygons of the same number of sides are similar.

14. A circle can be circumscribed about a parallelogram.

15. Two triangles are congruent if S. S. $\angle$ = S. S. $\angle$.

16. An equilateral triangle can be constructed if we know the radius of the inscribed circle.

17. Every diagonal of a regular polygon is greater than a side of the polygon.

18. The area of a rectangle is equal to one-half the product of its diagonals.

19. The perimeter of a square is twice the length of a diagonal of the square.

20. Two rectangles are similar polygons.

SELECTION TEST

EXERCISE 65

Underscore the word or symbol in each parenthesis that completes the statement correctly.

1. "At a given point in a given line one and only one line can be drawn perpendicular to the given line" is a (*definition, postulate, proposition*).

2. "Through two given points one and only one straight line can be drawn" is a (*definition, postulate, proposition*).

3. "If two triangles are mutually equiangular they are similar" is a (*definition, postulate, proposition*).

4. "An equilateral parallelogram is a rhombus" is a (*definition, postulate, proposition*).

5. If two adjacent angles have their exterior sides in the same straight line they are (*equal, right, supplementary, complementary*).

6. The area of a circle is ($2\pi r$, πd^2, $\frac{1}{2}\pi d^2$, πr^2).

7. The area of an equilateral triangle in terms of a side b is

$$\left(\frac{b^2}{2}, \frac{b^2}{2}\sqrt{3}, \frac{b^2}{2}\sqrt{2}, \frac{b^2}{4}\sqrt{3}\right)$$

8. Similar polygons have equal (*perimeters, areas, angles*).

9. A quadrilateral is a rhombus if (*its diagonals, a pair of angles, its sides*) are equal.

10. The area of a square is equal to the (*product of its diagonals, square of half its perimeter, product of two of its sides*).

11. A triangle is a right triangle if its sides are (2, 3, 4; 3, 4, 6; 1.5, 2.5, 2; 8, 12, 14).

12. If a quadrilateral is inscribed in a circle its opposite (*angles, sides*) are (*equal, complementary, supplementary, parallel*).

13. If the opposite angles of a quadrilateral are equal the figure is a (*trapezoid, square, parallelogram, rectangle*).

14. Through two points (*one, two, three, an unlimited number of*) circle(s) can be drawn.

15. An apothem of a regular polygon is the radius of the (*inscribed circle, circumscribed circle*).

16. A side of an inscribed regular hexagon is (*greater than, equal to, less than*) the radius of the circumscribed circle.

17. The area of a triangle whose sides are 9, 10, and 17 is (40, 28, 36, 32).

18. A hexagon has (6, 8, 9, 10) diagonals.

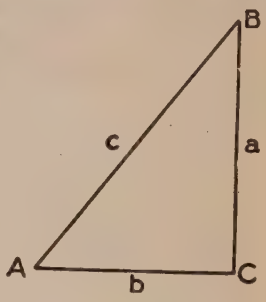
19. The areas of two similar triangles vary as (*their perimeters, the sum of their angles, the square of two corresponding altitudes*).

20. A circle can be circumscribed about any (*quadrilateral, isosceles trapezoid, parallelogram*).

TRIGONOMETRIC FUNCTIONS

SUMMARY

The standard method of lettering a right triangle is to denote the right angle by C, the two acute angles by A and B, the hypotenuse by c, and the sides opposite A and B by a and b respectively.



$$\sin A = \frac{a}{c} \qquad \cos A = \frac{b}{c} \qquad \tan A = \frac{a}{b}$$

$$\sin B = \frac{b}{c} \qquad \cos B = \frac{a}{c} \qquad \tan B = \frac{b}{a}$$

EXERCISE 66

1. In right triangle ABC, $a = 15$, $b = 20$, and $c = 25$. Fill in the decimal values of the functions of A and B in the following table.

	sin	cos	tan
A			
B			

2. In right triangle ABC, if $a = 12$ and $b = 5$, what is the value of sin A? of cos B?

3. By making use of an isosceles right triangle, find in radical form the values of the functions of 45°.

4. By making use of an equilateral triangle, find in radical form the values of the functions of (*a*) 60°; (*b*) 30°.

5. If an acute angle increases in value, do all of its trigonometric functions also increase in value? Explain.

6. (*a*) "Angle of elevation" or "angle of depression" means the angle formed by the line of sight and a ..... line. (*b*) Draw a figure and mark at point A the angle of depression of a lower point B. Mark also at B the angle of elevation of A. Why are these two angles equal?

7. (*a*) Compare the values of sin 50° and cos 40°. Which is greater? (*b*) Two other angles whose sine and cosine respectively

are related in the same way as those in (*a*) are ..... degrees and ..... degrees. (*c*) Is there an angle whose sine and cosine are equal?

8. Are the following statements true or false?

(*a*) The sine of 80° is twice as great as the sine of 40°.

(*b*) The cosine of 80° is less than the cosine of 40°.

9. If $\sin A = 0.4695$, find the value of A by using the table.

10. (*a*) Find B to the nearest degree, if $\cos B = 0.5206$. (*b*) Given $\tan A = 2.0204$, what is the value of A to the nearest degree?

SOLUTION OF RIGHT TRIANGLES

EXERCISE 67

Use of Sine

1. In right triangle ABC, find the length of b if $c = 18$ and $B = 48°$.

2. In right triangle ABC, find the length of c if $b = 38$ in. and $B = 73°$.

3. In right triangle ABC, $b = 3$ and $c = 4$. Find the value of B to the nearest degree.

4. (*a*) In the triangle of question 3, how many degrees are there in A? (*b*) Outline three ways of finding the length of a in the same triangle.

5. A telegraph pole is braced by a guy-wire fastened to the pole at a point 25 ft. from the ground. Find the length of the guy-wire if it makes an angle of 40° with the ground and an allowance of 3 ft. is made for fastening.

6. In rectangle $ABCD$, diagonal BD is drawn. If BC is 6 ft. and $\angle BDC$ is 36°, what is the length of diagonal BD?

7. Given triangle ABC (not a right triangle), with $AB = 10$ ft., $AC = 22$ ft., and $\angle A = 72°$. Find (*a*) the length of the altitude to AC, (*b*) the area of the triangle.

Use of Cosine

8. In right triangle ABC, find the length of a if $c = 24.8$ ft. and $B = 82°$.

9. (*a*) In right triangle ABC, find the length of c if $a = 7$ ft. 3 in. and $B = 63°$. (*b*) What would be the length of c if B were 60° instead of 63°?

10. In right triangle ABC, $b = 6$ and $c = 18$. Find the value of A to the nearest degree.

11. How far from a house must the foot of a ladder 25 ft. long be placed so that it will make an angle of 70° with the ground?

12. A ship sails in a direction 36° east of north. After the ship has sailed 50 miles on her course, how many miles farther north is she?

Use of Tangent

13. In right triangle ABC, find the length of b if $a = 16.41$ ft. and $B = 28°$.

14. In right triangle ABC, find the length of a if $b = 3.4$ ft. and $B = 39°$.

15. In right triangle ABC, find the value of A to the nearest degree if $a = 3$ ft. and $b = 2$ ft.

16. A ship captain finds from his reference book that a certain lighthouse is 160 ft. high. By using his sextant on the ship, he finds the angle of elevation of the top of the lighthouse to be 5°. How far is the ship from the lighthouse?

17. From the middle of a river two miles wide, the angle of elevation of the top of a cliff on the bank is found to be 6°. How high is the cliff?

18. To obtain the width of a river, a point B is selected on the opposite bank, and a line CA, 200 ft. long, is laid off so that $\angle ACB$ is 90°. By measurement, $\angle CAB$ is found to be 58°. How wide is the river?

Miscellaneous

19. When the angle of elevation of the sun is 70°, the shadow of a certain building is found to be 30 ft. long. Find the height of the building.

20. In parallelogram $ABCD$, $AB = 60$ ft., $BC = 100$ ft., and $\angle B = 65°$. Find the area of the parallelogram.

21. Find the altitude of an equilateral triangle whose side is 38 ft.

22. Find the side of an equilateral triangle whose altitude is 12 ft.

23. (*a*) What must be the length of a guy-wire which will just reach from the top of a 24-ft. telephone pole to a point on the ground 18 ft. from the pole? (*b*) What angle will the guy-wire make with the ground?

24. A hill has a slope of approximately 18°. From the foot of the hill to the top is 500 ft. What is the approximate height of the hill?

25. In right triangle ABC, $AC = 2.3$ ft. and $BC = 3.6$ ft. Find (*a*) the number of degrees in $\angle A$. (*b*) the length of the altitude to AB.

26. A regular octagon is inscribed in a circle whose radius is 6 in. (*a*) How many degrees are there in a central angle of the octagon? (*b*) Find the length of one side of the octagon, and also

the length of its perimeter. (*c*) Find the length of the apothem and also the area of the octagon.

27. Using a regular pentagon inscribed in a circle whose radius is 10 in., perform the same calculations as in **26.**

28. At a distance of 112 ft. from a building, the angle of elevation of the top of a flagpole on the building is 32°. From the same point, the angle of elevation of the bottom of the flagpole is 24°. Find the height of the flagpole.

29. Two radio compass stations A and B take the bearings of an "S O S" signal from a ship in distress. Station A is located 32 miles directly north of B. Station B finds the bearing of the ship to be north 58° east, and station A finds it to be south 42° east. (*a*) Which station is nearer to the ship? (*b*) Find the distance of B from a line joining A and the ship. (*c*) Find the distance of B from the ship.

30. Are the following statements true or false? (*a*) The sine of an angle cannot be greater than 1. (*b*) $\tan 45° = 1$. (*c*) $\tan A = \sin A \div \cos A$.

Tables of Natural Trigonometric Functions

Angle	Sine	Cosine	Tangent
1°	.0175	.9998	.0175
2°	.0349	.9994	.0349
3°	.0523	.9986	.0524
4°	.0698	.9976	.0699
5°	.0872	.9962	.0875
6°	.1045	.9945	.1051
7°	.1219	.9925	.1228
8°	.1392	.9903	.1405
9°	.1564	.9877	.1584
10°	.1736	.9848	.1763
11°	.1908	.9816	.1944
12°	.2079	.9781	.2126
13°	.2250	.9744	.2309
14°	.2419	.9703	.2493
15°	.2588	.9659	.2679
16°	.2756	.9613	.2867
17°	.2924	.9563	.3057
18°	.3090	.9511	.3249
19°	.3256	.9455	.3443
20°	.3420	.9397	.3640
21°	.3584	.9336	.3839
22°	.3746	.9272	.4040
23°	.3907	.9205	.4245
24°	.4067	.9135	.4452
25°	.4226	.9063	.4663
26°	.4384	.8988	.4877
27°	.4540	.8910	.5095
28°	.4695	.8829	.5317
29°	.4848	.8746	.5543
30°	.5000	.8660	.5774
31°	.5150	.8572	.6009
32°	.5299	.8480	.6249
33°	.5446	.8387	.6494
34°	.5592	.8290	.6745
35°	.5736	.8192	.7002
36°	.5878	.8090	.7265
37°	.6018	.7986	.7536
38°	.6157	.7880	.7813
39°	.6293	.7771	.8098
40°	.6428	.7660	.8391
41°	.6561	.7547	.8693
42°	.6691	.7431	.9004
43°	.6820	.7314	.9325
44°	.6947	.7193	.9657
45°	.7071	.7071	1.0000

Angle	Sine	Cosine	Tangent
46°	.7193	.6947	1.0355
47°	.7314	.6820	1.0724
48°	.7431	.6691	1.1106
49°	.7547	.6561	1.1504
50°	.7660	.6428	1.1918
51°	.7771	.6293	1.2349
52°	.7880	.6157	1.2799
53°	.7986	.6018	1.3270
54°	.8090	.5878	1.3764
55°	.8192	.5736	1.4281
56°	.8290	.5592	1.4826
57°	.8387	.5446	1.5399
58°	.8480	.5299	1.6003
59°	.8572	.5150	1.6643
60°	.8660	.5000	1.7321
61°	.8746	.4848	1.8040
62°	.8829	.4695	1.8807
63°	.8910	.4540	1.9626
64°	.8988	.4384	2.0503
65°	.9063	.4226	2.1445
66°	.9135	.4067	2.2460
67°	.9205	.3907	2.3559
68°	.9272	.3746	2.4751
69°	.9336	.3584	2 6051
70°	.9397	.3420	2.7475
71°	.9455	.3256	2.9042
72°	.9511	.3090	3.0777
73°	.9563	.2924	3.2709
74°	.9613	.2756	3.4874
75°	.9659	.2588	3.7321
76°	.9703	.2419	4.0108
77°	.9744	.2250	4.3315
78°	.9781	.2079	4.7046
79°	.9816	.1908	5.1446
80°	.9848	.1736	5.6713
81°	.9877	.1564	6.3138
82°	.9903	.1392	7.1154
83°	.9925	.1219	8.1443
84°	.9945	.1045	9.5144
85°	.9962	.0872	11.4301
86°	.9976	.0698	14.3007
87°	.9986	.0523	19.0811
88°	.9994	.0349	28.6363
89°	.9998	.0175	57.2900
90°	1.0000	.0000	

PLANE GEOMETRY

Tuesday, June 17, 1941 — 9.15 a. m. to 12.15 p. m., only

Part I

Answer all questions in this part. Each correct answer will receive 2 credits. No partial credit will be allowed. Each answer must be reduced to its simplest form.

Directions (questions 1-11)—Write on the dotted line at the right of each question the expression which, when inserted in the corresponding blank, will make the statement true.

1. The sum of an angle and its supplement is ____ 1.____

2. The formula for the perimeter P of a regular polygon in terms of the number of sides n and the length of each side s is $P =$ ____ 2.____

3. The area of a rhombus whose diagonals are 5 inches and 12 inches is ____ square inches. 3.____

4. In a right triangle the cosine of an acute angle is the ratio of the ____ side to the hypotenuse. 4.____

5. Triangle ABC is inscribed in a circle. Side AB is a diameter of the circle and arc AC equals 100°. The acute angle formed by the tangent at B and side BC of the triangle contains ____ degrees. 5.____

6. A line parallel to side AB of triangle ABC intersects CA at D and CB at E. If $CD = 4$, $DA = 2$ and $BC = 9$, then $CE =$ ____. 6.____

7. In the right triangle ABC, C is the right angle and CD is the altitude upon AB. If $AD = 4$ and $DB = 12$, then $AC =$ ____. 7.____

8. In triangle ABC, $AB = 10$ and angle $A = 30°$; then the length of the altitude upon AC is ____. 8.____

9. From a point P outside circle O, tangent PD and secant PCA are drawn. If tangent $PD = 8$ and $PC = 4$, then secant $PCA =$ ____. 9.____

10. If the areas of two similar triangles are in the ratio 1 : 25, then any two corresponding sides of the triangles are in the ratio ____. 10.____

11. In the construction of a triangle congruent with a given triangle it is necessary to use three parts of the given triangle, one of which must be ____. 11.____

12. Find the area of a trapezoid whose bases are 7 and 12 and whose altitude is 6. 12.____

13. Find, correct to the *nearest tenth*, the circumference of a circle whose radius is 4. [Use $\pi = 3.14$] 13_____

Directions (questions 14-18)—Indicate the correct answer to each of the questions by writing on the dotted line at the right the letter *a*, *b* or *c*.

14. Two consecutive angles of a parallelogram are always (*a*) equal, (*b*) complementary or (*c*) supplementary. 14_____

15. The median of an equilateral triangle is (*a*) greater than the altitude, (*b*) equal to the altitude or (*c*) less than the altitude. 15_____

16. The expression for the area A of a circle in terms of its diameter d is (*a*) πd^2, (*b*) $\frac{1}{4}\pi d^2$ or (*c*) $\frac{1}{2}\pi d^2$. 16_____

17. In triangle ABC, angle B is 60° and AB is greater than AC; then angle C is (*a*) equal to angle A, (*b*) greater than angle A or (*c*) less than angle A. 17_____

18. The locus of the centers of equal circles passing through a given point is (*a*) a point, (*b*) a straight line or (*c*) a circle. 18_____

Directions (questions 19-23)—Indicate whether each statement is *always* true, *sometimes* true or *never* true by writing on the line at the right the word *always, sometimes* or *never.*

19. If the three altitudes of a triangle are equal, the triangle is equilateral. 19_____

20. Two polygons are congruent if their corresponding sides are equal. 20_____

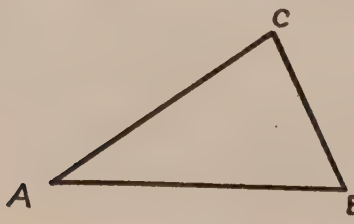
21. If two chords of a circle are parallel, the minor arc of one chord is equal to the minor arc of the other. 21_____

22. When two parallel lines are cut by a transversal which is not perpendicular to them, all the acute angles formed are equal. 22_____

23. Two triangles are congruent if two sides and an angle of one are equal respectively to two sides and an angle of the other. 23_____

Directions (questions 24-25)—Leave all construction lines on the paper.

24. On line segment a corresponding to side AB of triangle ABC, construct a triangle similar to triangle ABC.



a

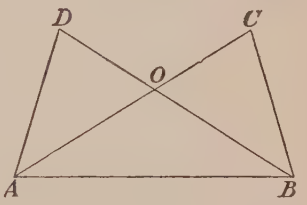
25. Construct a circle of given radius r which shall be tangent to the given line m at the given point P.

Part II

Answer two questions from this part.

26. Prove that if two chords intersect within a circle, the product of the segments of one is equal to the product of the segments of the other. [10]

27. In triangles ABC and ABD, $BC = AD$, $AC = BD$, and AC and BD intersect at O.
Prove: $AO = BO$ [10]



28. An angle inscribed in a circle is measured by one half its intercepted arc.

a. Prove the above theorem for the case in which one side of the angle is a diameter. [8]

b. Which one of the following axioms would be used in proving this theorem for the case in which the sides of the angle lie on opposite sides of the center? [2]

(1) If equals are added to equals, the sums are equal.

(2) If equals are subtracted from equals, the remainders are equal.

Part III

Answer two questions from this part.

29. In parallelogram $ABCD$, angle A is 45°, altitude DE on base AB is 8 and diagonal DB is 17.

a. Find AE. [2]

b. Find EB. [4]

c. Find the area of the parallelogram. [4]

30. A circle is inscribed in a regular pentagon. If a side of the pentagon is 50, find, correct to the *nearest integer*, the radius of the circle. [10]

31. In trapezoid $ABCD$, AB is the lower base. The bisector of angle A intersects DC at P.

a. Prove that triangle ADP is isosceles. [5]

b. If $AD = 8$ and the altitude of the trapezoid is 6, find the area of triangle ADP. [5]

Part IV

Answer one question from this part.

32. For *each* of the following statements write a converse and indicate whether the converse is true or false:

a. If one of the altitudes of a triangle bisects the opposite side, the triangle is isosceles. [2, 1]

b. If a given polygon is regular, then a circle can be circumscribed about that polygon. [2, 1]

c. If the fathers of two boys are brothers, then the two boys are first cousins. [1, 1]

d. If in a certain school a pupil's average grade is at least 90 per cent, his name is placed on the scholastic honor roll. [1, 1]

33. Triangle ABC is inscribed in a circle and a tangent is drawn at A. A line parallel to the tangent intersects AB at R and AC at S. Prove: $AR \times AB = AC \times AS$ [10]

PLANE GEOMETRY

Tuesday, January 20, 1942—9.15 a. m. to 12.15 p. m., only

Part I

Answer all questions in this part. Each correct answer will receive 2 credits. No partial credit will be allowed. Each answer must be reduced to its simplest form.

Directions (questions 1-13)—Write on the dotted line at the right of each question the expression which when inserted in the corresponding blank will make the statement true.

1. If two lines are cut by a transversal making two alternate interior angles equal, the sum of the two interior angles on the same side of the transversal is ..... degrees. 1.....

2. If two tangents from an external point to a circle intercept a major arc of 210°, the angle between the tangents contains ..... degrees. 2.....

3. From a point P outside a circle, a secant PCA is drawn. If PC equals 4 inches and PCA equals 16 inches, then the length of a tangent drawn to the circle from P is ..... inches. 3.....

4. If the sides of two regular hexagons are in the ratio 1:3, their areas are in the ratio ...... 4.....

5. In right triangle ABC, C is the right angle and CD is the altitude on AB. If $AB = 9$ and $AC = 6$, then $AD =$ ..... 5.....

6. If an exterior angle of an equiangular polygon is 30°, the number of sides of the polygon is ...... 6.....

7. The altitude of an equilateral triangle whose side is 4 is ..... [Answer may be left in radical form.] 7.....

8. If $\cos x = \sin 30°$, angle x contains ..... degrees. 8.....

9. D is the mid-point of side AB of triangle ABC, and a line from D parallel to AC cuts BC in E. If $AC = 9$ inches, then $DE =$ ..... inches. 9.....

10. If in triangle ABC, angle $C = 90°$, $AB = 30$ and $BC = 15$, then angle B contains ..... degrees. 10.....

11. If the circumference of a circle is 14π, the area of this circle in terms of π is ..... 11.....

12. The area of a regular polygon whose perimeter is 80 and whose apothem is 12 is ..... 12.....

13. If the area of a sector of a circle is to the area of the circle as 1:8, then the number of degrees in the angle of the sector is ..... 13.....

Directions (questions 14-18)—Indicate the correct answer to each question by writing on the dotted line at the right the letter *a*, *b* or *c*.

14. If triangle ABC is divided into two congruent triangles by the median drawn from vertex C, then triangle ABC must be (*a*) isosceles, (*b*) equilateral or (*c*) right. 14.....

15. If the diagonals of a quadrilateral bisect each other, the quadrilateral is always a (*a*) rhombus, (*b*) rectangle or (*c*) parallelogram. 15_____

16. The locus of points equidistant from the four vertices of a given rectangle is (*a*) a pair of lines, (*b*) one line or (*c*) a point. 16_____

17. If in a rhombus $ABCD$, angle $A = 110°$, then (*a*) $AC = BD$, (*b*) $AC > BD$ or (*c*) $AC < BD$. 17_____

18. As the number of sides of a regular polygon inscribed in a given circle increases, (*a*) the central angle increases, (*b*) the apothem increases or (*c*) each exterior angle increases. 18_____

Directions (questions 19-23)—Indicate whether each of the following statements is *always* true, *sometimes* true or *never true* by writing the word *always, sometimes* or *never* on the dotted line at the right.

19. The diagonals of a quadrilateral divide it into four congruent triangles. 19_____

20. The center of a circle circumscribing a triangle is inside the triangle. 20_____

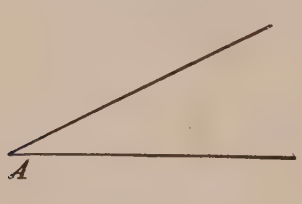
21. If the sum of two angles of a triangle is equal to the third angle, then two sides of the triangle are perpendicular to each other. 21_____

22. The bisectors of a pair of adjacent supplementary angles are perpendicular to each other. 22_____

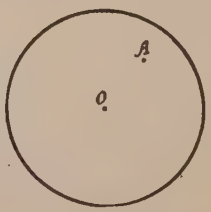
23. A straight line parallel to one of two intersecting lines is parallel to the other also. 23_____

Directions (questions 24-25)—Leave all construction lines on the paper.

24. If angles A and B are two angles of a triangle, find by construction the third angle of the triangle.



25. Through point A, construct the chord of circle O that is bisected by A.



Part II

Answer two questions from this part.

26. Prove that the area of a parallelogram is equal to the product of its base and its altitude. [10]

27. AB is a diameter of a circle and C is any point on the circle. Chord AC is drawn and extended to D so that $AC = CD$, and DB is drawn. Prove that $DB = AB$. [10]

28. Prove that if two triangles have an angle of one equal to an angle of the other and the sides including these angles proportional, the triangles are similar. [10]

Part III

Answer two questions from this part.

29. The base of a triangle is 30 inches and the altitude on this base is 15 inches. Find the area of the trapezoid formed by a line parallel to the base and 9 inches from the opposite vertex. [10]

30. The altitude on the hypotenuse of a right triangle divides the hypotenuse into segments of 9 and 4. Find, correct to the *nearest degree*, the smaller acute angle of the original triangle. [10]

31. In triangle ABC, angle $C = 90°$, CM is the median to AB and MD is perpendicular to CB. If $CM = 10$ and $MD = 8$, find the the perimeter of triangle ABC. [10]

Part IV

Answer one question from this part.

32. For *each* of the following statements indicate whether you have been given too little information, just enough information or more information than is needed to justify the conclusion:

a. If A is 10 miles from B and B is 15 miles from C, then A is 25 miles from C. [2]

b. If the length of its apothem and the length of its radius are given, a regular pentagon can be constructed. [2]

c. If two boys are first cousins, then their fathers are brothers. [2]

d. If the perimeter and a diagonal of a rectangle are known, then the dimensions of the rectangle can be found. [2]

e. A arrived in Chicago 10 hours after a theft was committed in New York. If the fastest train from New York to Chicago makes the trip in 16 hours, then A did not commit the theft. [2]

33. Prove that if the three sides of a triangle are unequal, the bisector of any angle of that triangle can not be perpendicular to the opposite side. [Suggestion: Use the indirect method.] [10]

PLANE GEOMETRY

Tuesday, June 16, 1942 — 9.15 a. m. to 12.15 p. m., only

Part I

Answer all questions in this part. Each correct answer will receive 2 credits. No partial credit will be allowed. Each answer must be reduced to its simplest form.

Directions (questions 1-12) — Write on the dotted line at the right of each question the expression which when inserted in the corresponding blank will make the statement true.

1. Diagonal AC divides parallelogram $ABCD$ into two triangles of which angle ACB and angle _____ are corresponding parts. 1_____

2. A regular hexagon is inscribed in a circle. If the radius of the circle is 2 inches, the perimeter of the hexagon is _____ inches. 2_____

3. If the diameter of a circle is 10, the area of the circle in terms of π is _____. 3_____

4. The sides of a given triangle are 6, 9 and 12 inches. The perimeter of a similar triangle whose shortest side is 4 inches is _____ inches. 4_____

5. If in right isosceles triangle ABC the bisector of the right angle C is 6 inches in length, then the hypotenuse AB is _____ inches. 5_____

6. A diameter of a circle is parallel to a chord that subtends an arc of 100°. One of the arcs intercepted between the diameter and the chord contains _____ degrees. 6_____

7. If the diagonals of a rhombus are 14 and 48, one side of the rhombus is _____. 7_____

8. The area of a trapezoid is 55 square inches. If the bases are 14 inches and 8 inches, then the altitude is _____ inches. 8_____

9. From a point outside a circle a tangent and a secant are drawn to the circle. If the secant is 27 inches and its external segment is 3 inches, the tangent is _____ inches. 9_____

10. If two adjacent sides of a parallelogram are 6 inches and 8 inches and the included angle is 30°, the area of the parallelogram is _____ square inches. 10_____

11. One of the arcs intercepted by two tangents drawn to a circle from an external point contains 260°. The number of degrees in the angle formed by the tangents is _____. 11_____

12. The sides of a right triangle are 3, 4 and 5. The smallest angle of this triangle, correct to the *nearest degree,* contains _____ degrees. 12_____

13. If two rectangles have equal perimeters, must they have equal areas? [Answer *yes* or *no*.]. 13._____

14. If the area of an equilateral triangle is $9\sqrt{3}$, what is the length of one side? 14._____

Directions (questions 15-19) — Indicate the correct answer to each question by writing on the dotted line at the right the letter *a*, *b* or *c*.

15. If two right triangles have an acute angle of one equal to an acute angle of the other, then the triangles must be (*a*) congruent, (*b*) similar, or (*c*) equal. 15._____

16. If the diagonals of a rhombus are 8 and 12, the area of the rhombus is (*a*) 24, (*b*) 48 or (*c*) 96. 16._____

17. A good citizen votes at election time. Jones votes at election time. Therefore Jones is a good citizen.

The above reasoning (*a*) is unsound and is an example of indirect reasoning, (*b*) is sound and is an example of direct reasoning or (*c*) is unsound and is an example of reasoning from a converse. 17._____

18. The locus of the center of a circle having a given line segment as a chord is (*a*) a line parallel to the given line segment, (*b*) a circle having the given line segment as a diameter, or (*c*) the perpendicular bisector of the given line segment. 18._____

19. If the circumference of a circle is C, the radius of this circle is (*a*) $\frac{C}{\pi}$, (*b*) $\frac{C}{2\pi}$ or (*c*) $\sqrt{\frac{C}{\pi}}$ 19._____

Directions (questions 20-23)—Indicate whether each statement is true or false by writing the word *true* or the word *false* on the dotted line at the right.

20. The altitudes of a triangle always intersect inside the triangle. 20._____

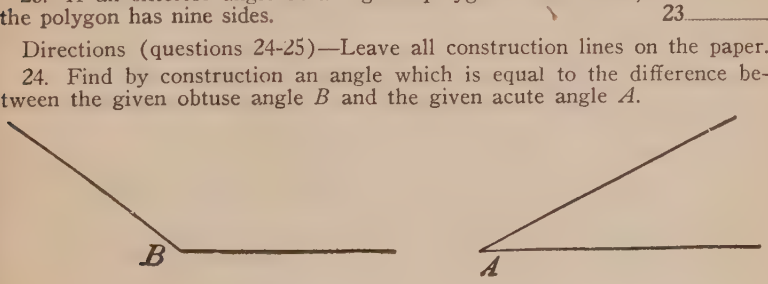
21. In triangle ABC, if AB is greater than AC, then angle C is greater than angle B. 21._____

22. If three sides of a triangle are unequal, the medians of the triangle will not pass through the same point. 22._____

23. If an exterior angle of a regular polygon contains 40°, the polygon has nine sides. 23._____

Directions (questions 24-25)—Leave all construction lines on the paper.

24. Find by construction an angle which is equal to the difference between the given obtuse angle B and the given acute angle A.



25. Construct an isosceles triangle whose base is the given line segment b and whose altitude upon b is the given line segment h.

b

h

Part II

Answer two questions from this part.

26. Prove that an angle formed by two chords intersecting within a circle is measured by one half the sum of the intercepted arcs. [10]

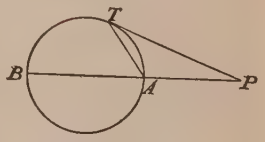
27. Prove that the diagonals of a trapezoid divide each other proportionally. [10]

28. *a.* Prove that any point which is equidistant from the sides of an angle is on the bisector of the angle. [7]
    b. Write the converse of the proposition stated in *a*. [2]
    c. Is either the proposition in *a* or its converse, taken alone, sufficient to warrant the conclusion that the locus of points equidistant from the sides of an angle is the bisector of the angle? [1]

Part III

Answer two questions from this part.

29. In the accompanying figure, PT is a tangent, PAB is a sectant and TA a chord. Arc $BT = 112°$ and angle $T = 34°$.



    a. Find the number of degrees in angle BAT. [2]
    b. Find the number of degrees in angle P. [4]
    c. Show that AB is a diameter of the circle. [4]

30. In a given circle the line segment joining the mid-point of a chord and the mid-point of its minor arc is 4 inches long. The chord is 16 inches long.
    a. Find the radius of the circle. [Show all work.] [7]
    b. Find the area of the triangle formed by the chord and the radii drawn to its end points. [3]

31. A regular pentagon whose side is 22.0 inches is inscribed in a circle.
    a. Find, correct to the *nearest tenth of an inch,* the apothem of the pentagon. [6]
    b. Using the result obtained in answer to *a*, find, correct to the *nearest square inch,* the area of the pentagon. [4]

Part IV

Answer one question from this part.

32. In each of the following propositions the hypothesis contains one or more facts which we do not need to know in order to prove the conclusion. In *each* case, merely write the fact or facts *not needed*.

a. If one base of a trapezoid is twice the other and the diagonals are equal, then the trapezoid is isosceles. [2]

b. If a leg and the adjacent acute angle of one right triangle are equal to the corresponding parts of another right triangle, then the triangles are congruent. [2]

c. In a quadrilateral having two sides equal and parallel, the line segments joining the mid-points of the four sides, taken in order, form a parallelogram. [2]

d. The areas of two similar triangles are to each other as the product of the base and the altitude of one is to the product of the base and the altitude of the other. [2]

e. If the sum of the exterior angles of a regular polygon made by producing each of its sides in succession is one half the sum of its interior angles, then the polygon has six sides. [2]

33. $ABCD$ is a parallelogram whose diagonals AC and BD intersect in O. Line segments AO, BO, CO and DO are represented by $3x + 2y$, $x + 18$, $7x - 6y$ and $5x - 2y$ respectively. Is $ABCD$ a rectangle? [Leave all work on your paper.] [10]

PLANE GEOMETRY

Tuesday, January 19, 1943 — 9.15 a. m. to 12.15 p. m., only

Part I

Answer all questions in this part. Each correct answer will receive 2 credits. No partial credit will be allowed. Each answer must be reduced to its simplest form.

Directions (questions 1-10) — Write on the line at the right of *each* question the expression which when inserted in the blank will make the statement true.

1. If two chords of a circle bisect each other, the chords must be _____ of the circle. 1_____

2. Chords AB and CD intersect within a circle at point E. If $AE = 3$, $EB = 12$ and $CE = 4$, then ED equals _____ 2_____

3. In a circle a chord 8 inches long is 3 inches from the center. The number of inches in the length of the radius is _____ 3_____

4. From a point P outside a circle, a secant PCA is drawn. If $PC = 5$ inches and $PA = 20$ inches, then the length of a tangent drawn to the circle from P is _____ inches. 4_____

5. The sides of a triangle are 5, 6 and 7 and the perimeter of a similar triangle is 36. The side of the second triangle corresponding to side 5 of the first is _____ 5_____

6. The area of a rhombus in terms of its diagonals d and d' is _____. 6_____

7. The area of an equilateral triangle whose perimeter is 12 is _____ [Answer may be left in radical form.] 7_____

8. Triangle ABC is inscribed in a circle. Point P bisects minor arc AB and chord BP is drawn. If angle ABP is 40 degrees, the number of degrees in angle C is _____ 8_____

9. An architect uses a scale of ¼ inch to a foot. If the dimensions of a room as represented on the plans of a house are 4¼ in. by 3¾ in., the actual dimensions in feet are _____ 9_____

10. If the angle of a sector is 30 degrees and the radius of the circle is 12, the area of the sector is _____ [Answer may be left in terms of π.] 10_____

Directions (questions 11-15) — Indicate the correct answer to *each* question by writing on the line at the right the letter *a*, *b* or *c*.

11. Two figures must be similar if they are (*a*) rectangles, (*b*) equilateral triangles or (*c*) rhombuses. 11_____

12. If two circles have a common chord, then the sum of their radii is (*a*) greater than, (*b*) equal to or (*c*) less than, the distance between their centers. 12_____

13. In triangle ABC, side CA is greater than side CB and angle B equals 35 degrees. Angle C is (*a*) an acute angle, (*b*) a right angle or (*c*) an obtuse angle. 13_____

14. If a median of a triangle is equal to one half the side to which it is drawn, the triangle is (*a*) acute, (*b*) obtuse or (*c*) right. 14_____

15. Below are given a proposition and one of its converses.

Proposition: The line which bisects the vertex angle of an isosceles triangle is perpendicular to the base and bisects the base.

Converse: The line from the vertex of an isosceles triangle which is perpendicular to the base bisects the base and also the vertex angle.

a. Both the proposition and the converse are true.
b. The proposition is true and the converse is false.
c. Both the proposition and the converse are false. 15._____

Directions (questions 16-23) — Indicate whether *each* statement is true or false by writing the word *true* or *false* on the line at the right.

16. The altitude upon any side of a triangle is greater than the median drawn to that side. 16._____

17. A parallelogram is defined as a polygon whose opposite sides are parallel. 17._____

18. Sectors of unequal circles may be congruent. 18._____

19. The perpendicular bisector of line segment AB is the locus of the centers of all circles passing through A and B. 19._____

20. If a secant and a tangent to a circle are parallel, the diameter drawn to the point of tangency is perpendicular to the secant. 20._____

21. If the altitude CD is drawn upon the hypotenuse AB of right triangle ABC, the angle ACD equals the angle CBD. 21._____

22. A diagonal of an isosceles trapezoid divides the trapezoid into two congruent triangles. 22._____

23. If the mid-points of the sides of a rhombus are joined in order, the figure thus formed is a rectangle. 23._____

Directions (questions 24-25) — Leave all construction lines on the paper.

24. Given line segment a and fixed point P. Construct the locus of points whose distance from P is $\frac{1}{2}a$.

a $\quad\quad$ P

25. Construct triangle RST in which angle R = angle A, $RS = c$ and $RT = b$

Part II

Answer two questions from part II.

26. Prove that if the opposite sides of a quadrilateral are equal the figure is a parallelogram. [10]

27. In triangle ABC, angle C is obtuse. AE is the altitude on BC and CD is the altitude on AB.

a. Prove that $BC : AB = CD : AE$ [8]
b. Would this proportion hold true if angle C were acute? [2]

28. Prove that the area of a trapezoid is equal to one half the product of its altitude and the sum of its bases. [10]

Part III

Answer two questions from part III.

29. Sides AB and BC of parallelogram $ABCD$ are 10 and 6 respectively. The bisector of angle A meets DC at point E and BC extended at point F.

a. Prove that triangle ADE is isosceles. [4]

b. Prove that triangle ADE is similar to triangle ECF. [3]

c. Find the length of CF. [3]

30. PA and PB are tangents to circle O and intersect at an angle of 60 degrees. Radius OA is 8 inches long.

a. Find, correct to the *nearest inch,* the length of AP. [6]

b. Find the length of minor arc AB. [Answer may be left in terms of π.] [4]

31. A regular pentagon is circumscribed about a circle whose radius is 10.

a. Find a side of the pentagon correct to the *nearest tenth.* [6]

b. Using the result obtained in answer to *a,* find the area of the pentagon correct to the *nearest integer.* [4]

Part IV

Answer one question from part IV.

32. For *each* of the following statements indicate whether you have been given too little information, just enough information or more information than is needed to justify the conclusion.

a. The areas of two isosceles triangles are to each other as the squares of any two corresponding sides. [2]

b. If two chords of a circle are equal and parallel, then the chord passing through their mid-points is a diameter. [2]

c. If the drive wheels on a locomotive are each 60 inches in diameter and each makes 168 revolutions per minute, then the speed of the locomotive can be determined. [2]

d. A boy said he had just 45 cents in his pocket, consisting of 5 coins one of which was a nickel. His friend concluded that one of the coins must be a quarter. [2]

e. The entrance requirements of a certain college are: an average grade of not less than 80% in science, mathematics and English, and a grade of not less than 90% in one of them. Therefore a candidate for admission who has an average of 80% in the required subjects and whose grade in English is 60% meets these requirements. [2]

33. Tangents PA and PB are drawn to a circle. Diameter AD and chords AB and DB are drawn. Let the number of degrees in minor arc AB be represented by x.

a. Express in terms of x the number of degrees in

(1) Major arc AB [1]
(2) Minor arc DB [1]
(3) Angle ADB [1]
(4) Angle APB [2]

b. As x decreases,

(1) Does angle APB increase, decrease or remain the same? [2]
(2) Does the ratio of angle DAB to angle APB increase, decrease or remain the same? [3]

PLANE GEOMETRY

Tuesday, June 15, 1943 — 9.15 a. m. to 12.15 p. m., only

Part I

Answer all questions in this part. Each correct answer will receive 2 credits. No partial credit will be allowed. Each answer must be reduced to its simplest form.

Directions (questions 1-3) — Indicate the correct answer to *each* question by writing on the line at the right the letter *a, b* or *c.*

1. The sum of the interior angles of a polygon is 1800 degrees. The polygon has (*a*) 8 sides, (*b*) 10 sides, (*c*) 12 sides. 1.....

2. If two altitudes of a given triangle fall outside the triangle, the triangle is (*a*) right, (*b*) acute, (*c*) obtuse 2.....

3. If the length of the apothem of a regular pentagon is *m*, then the length of half of one side of the pentagon is (*a*) *m* cos 36°, (*b*) *m* sin 36°, (*c*) *m* tan 36° 3.....

Directions (questions 4-9) — If the blank in each statement is replaced by one of the words *always, sometimes* or *never,* the resulting statement is true. Select the word that will correctly complete *each* statement and write this word on the line at the right.

4. A parallelogram whose diagonals are equal and perpendicular is ..... a square. 4.....

5. The perpendicular bisector of a chord of a circle ..... passes through the center of the circle. 5.....

6. If two isosceles triangles have a side of one equal to the corresponding side of the other, the triangles are ..... congruent. 6.....

7. If the sides of one triangle are parallel to the sides of another triangle, the triangles are..... similar. 7.....

8. All three exterior angles of a triangle are ..... acute. 8.....

9. If two sides of a quadrilateral are parallel and the other two sides are equal, the quadrilateral is ..... a parallelogram. 9.....

Directions (questions 10-23) — Write the answer to *each* question on the line at the right.

10. The bases of a trapezoid are 8 and 12 and its area is 140. What is its altitude? 10.....

11. The three sides of a triangle are 8″, 10″ and 12″. What is the length of the line segment drawn from the mid-point of the 8″ side to the mid-point of the 12″ side? 11.....

12. A vertical pole 10′ high casts a shadow 8′ long and at the same time a near-by tree casts a shadow 40′ long. What is the height of the tree? 12.....

13. The sides of an angle, formed by a tangent and a secant drawn to a circle from an external point, intercept arcs of 140° and 30°. How many degrees are there in the angle? 13.....

14. Quadrilateral $ABCD$ is inscribed in a circle. If angle A contains 100 degrees, find the number of degrees in angle C. 14_____

15. If the radius of a circle is doubled, by what number is the circumference of the circle multiplied? 15_____

16. A regluar hexagon is inscribed in a circle and a tangent is drawn to the circle at a vertex of the hexagon. How many degrees are there in one of the acute angles thus formed? 16_____

17. One acute angle of a right triangle is double the other. The longer leg is $5\sqrt{3}$. How long is the hypotenuse? 17_____

18. An arc of a circle contains 72° and is 10 inches long. Find the circumference of the circle. 18_____

19. The area of the sector of a circle whose angle is 40° is 4π. Find the radius of the circle. 19_____

20. If two polygons are similar and a side of one is two thirds as long as the corresponding side of the other, what is the ratio of the area of the smaller polygon to the area of the larger polygon? 20_____

21. The legs of an isosceles triangle are AC and AB, and AD is a median. $AC = 5$ and $AD = 4$. Find the length of the base BC. 21_____

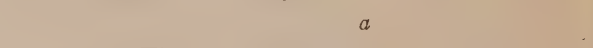
22. How many points are there which are 2 inches from a given line and 3 inches from a given point in that line? 22_____

23. Triangle ABC is a right triangle with right angle at C and E is any point on side AC. Line ED is perpendicular to AB at D. Complete the proportion $DE : BC = DA :$ _____ 23_____

Directions (questions 24-25) — Leave all construction lines on the paper.

24. At point P construct a line making an angle of 30° with line m.

25. Divide line segment a into *three* equal parts.



Part II

Answer two questions from part II.

26. Prove than in an isosceles triangle the angles opposite the equal sides are equal. [10]

27. On diameter AB of semicircle O two points, C and D, are located on opposite sides of the center O so that $AC = BD$. At C and D perpendiculars are erected to AB and extended to meet the semicircle in points E and F respectively. Prove that CE equals DF. [10]

28. Prove that two triangles are similar if an angle of one triangle is equal to an angle of the other and the sides including the angles are in proportion. [10]

Part III

Answer two questions from part III.

29. Given a circle whose radius is 4; a chord of this circle is drawn perpendicular to and bisecting one radius.

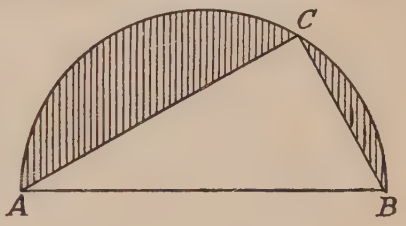
a. How many degrees are there in the minor arc of this chord? [4]

b. Find the length of this arc to the *nearest integer.*
[Use $\pi = 3.14$] [3]

c. Find the length of the chord correct to the *nearest integer.*.....[3]

30. The diagonals of a rectangle are each 22 and intersect at an angle of 110°. Find, correct to the *nearest integer,* the sides of the rectangle. [Use numerical trigonometry.] [6, 4]

31. In the semicircle at the right, chord $AC = 8$ and chord $BC = 6$. Find the area of the shaded portion correct to the *nearest integer.* [Use $\pi = 3.14$] [10]



Part IV

Answer one question from part IV.

32. AB is a given line segment which is fixed in position. Triangles are drawn having AB as one side and C as the vertex opposite side AB. Describe the locus of vertex C if

a. Triangle ABC is isosceles [2]

b. Angle C is a right angle [2]

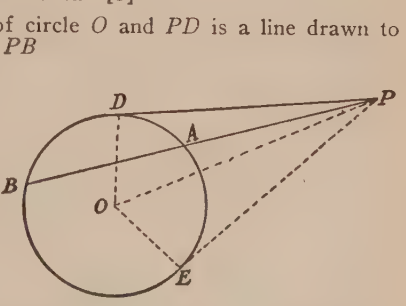
c. The median upon AB is a given line segment [3]

d. Triangle ABC has a constant area [3]

33. *a.* Given: PAB is a secant of circle O and PD is a line drawn to circle O such that $(PD)^2 = PA \times PB$

Prove: PD is tangent to circle O.

Suggestions: Draw PE tangent to circle O at E; also draw radii OE and OD and line OP. Then prove
(1) $PE = PD$ [3]
(2) $\triangle POD \cong \triangle POE$
(3) PD is tangent to circle O [4]



b. State the proposition of which this exercise is a converse. [1]

PLANE GEOMETRY

Tuesday, January 18, 1944 — 9.15 a. m. to 12.15 p. m., only

Part I

Answer all questions in this part. Each correct answer will receive 2 credits. No partial credit will be allowed. Each answer must be reduced to its simplest form.

1. In parallelogram $ABCD$, angle B is twice as large as angle A. How many degrees are there in angle A? 1_____

2. In right triangle ABC, altitude CD is drawn to hypotenuse AB. If angle A contains 32°, how many degrees are there in angle BCD? 2_____

3. How many degrees are there in each angle of a regular polygon of 8 sides? 3_____

4. Angle ABC, formed by diameter AB and chord BC of a circle, contains 30°. If the diameter of the circle is 10, find chord AC. 4_____

5. A line parallel to base AB of triangle ABC intersects the sides AC and BC at points D and E respectively. If $CD = 4$, $DA = 6$ and $CE = 8$, find EB. 5_____

6. The area of a triangle is 24. If one side of the triangle is 12, find the altitude drawn to that side. 6_____

7. Find the ratio of the areas of two similar polygons whose corresponding sides are in the ratio 2:3. 7_____

8. The area of a circle is 49π. Find its circumference. [Answer may be left in terms of π.] 8_____

9. Find the area of a square circumscribed about a circle whose radius is 10. 9_____

10. In triangle ABC, angle $C = 90°$, $BC = 3$, $AC = 4$; find angle A correct to the *nearest degree.* 10_____

Directions (questions 11–17) — If the blank in each statement is filled by one of the words *always, sometimes* or *never,* the resulting statement will be true. Select the word that will correctly complete *each* statement and write the word on the line at the right.

11. Triangles that have equal bases and equal altitudes are _____ congruent. 11_____

12. If angle B of triangle ABC contains 95°, then it is _____ possible to determine which is the longest side of the triangle. 12_____

13. The altitude and the median are drawn from the same vertex of a triangle. The altitude is _____ greater than the median. 13_____

14. The center of the circle which is circumscribed about a triangle _____ lies inside the triangle. 14_____

15. If the perpendiculars drawn from the center of a circle upon two chords are equal, then the minor arcs subtended by these chords are _____ equal. 15_____

16. Tangents drawn from an external point to a circle ____ make equal angles with the chord joining the points of tangency. 16.....

17. If the dimensions of a rectangle are doubled, then its area is ____ doubled. 17.....

Directions (questions 18–23) — Indicate the correct answer to *each* question by writing on the line at the right the letter *a*, *b*, *c* or *d*.

18. Two angles that are both equal and supplementary are (*a*) adjacent, (*b*) acute, (*c*) right, (*d*) obtuse. 18.....

19. The mid-point of the hypotenuse of a right triangle is the point of intersection of (*a*) the three medians, (*b*) the bisectors of the three angles, (*c*) the perpendicular bisectors of the three sides, (*d*) the three altitudes. 19.....

20. If the bases of a trapezoid are b and b', then the median of the trapezoid, in terms of b and b', is (*a*) $\frac{1}{2}\ bb'$, (*b*) $\frac{1}{2}(b + b')$, (*c*) $b + b'$, (*d*) $2(b + b')$ 20.....

21. If acute triangle ABC is inscribed in a circle and angle B is greater than agle A, then arc BC is (*a*) greater than arc AC, (*b*) equal to arc AC, (*c*) less than arc AC. 21.....

22. The locus of the center of a circle having a given line segment as a chord is (*a*) a line parallel to the chord, (*b*) the mid-point of the chord, (*c*) the perpendicular bisector of the chord, (*d*) a circle with the chord as diameter. 22.....

23. The area of a sector of n degrees in a circle of radius r is

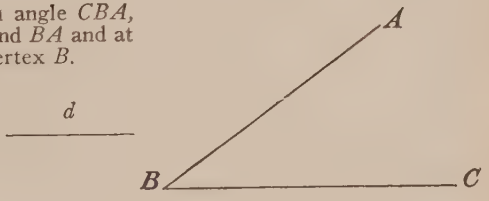
(*a*) $\frac{n}{360} \times \pi r^2$, (*b*) $\frac{n}{180} \times \pi r^2$, (*c*) $\frac{n}{180} \times \pi r$, (d) $\frac{n}{360} \times \pi d^2$

23.....

Directions (questions 24–25) — Leave all construction lines on the paper.

24. Construct a circle of radius r that will be tangent to line m at point P.

25. Find the point within angle CBA, equidistant from sides BC and BA and at the given distance d from vertex B.



Part II

Answer two questions from part II.

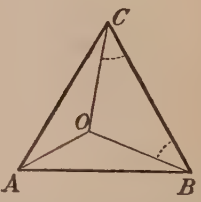
26. *a.* Prove that the sum of the interior angles of a triangle is a straight angle. [6]

b. Prove that the two acute angles of a right triangle are complementary. [4]

27. *a.* For the following exercise, draw a figure, letter it and in terms of the figure state what is given and what is to be proved: [No proof is required in this part.] [3]

From any point in the base of an isosceles triangle perpendiculars are drawn to the equal sides. Prove that the sum of these perpendiculars is equal to the altitude drawn to one of the equal sides.

b. Given triangle ABC with $AB = BC = CA$ and angle OCB = angle OBC Prove that AO bisects *angle* BAC. [7]



28. Prove that if from a point outside a circle a tangent and a secant are drawn to the circle, the tangent is the mean proportional between the secant and its external segment. [10]

Part III

Answer two questions from part III.

29. Nine circular disks each 3 inches in diameter were cut from a square of aluminum alloy 9 inches on a side. How much of the metal was wasted? Give your answer correct to the *nearest square inch.* [Use $\pi = \frac{22}{7}$] [10]

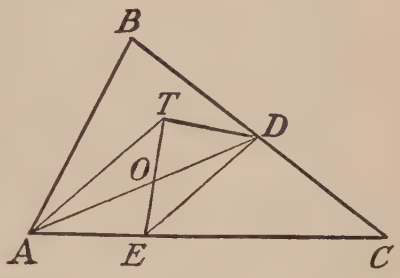
30. Tangent PC and secant PBA are drawn from point P to a circle and chords BC and AC are drawn. Angle $APC = 37°$ and arc $ABC = 236°$. Find the number of degrees in arc BC, angle BCP and angle ACB. [5, 2, 3]

31. The base of a parallelogram is 16 and one base angle is 36°. If the area of the parallelogram is 80, find, correct to the *nearest tenth,* the side adjacent to the base. [10]

Part IV

Answer one question from part IV.

32. T is any point in triangle ABC and D is the mid-point of BC. Points T and A are joined and DE is drawn parallel to TA. Lines TD, TE and AD are drawn.



Prove:

a. Triangle ADC = triangle ADB [3]

b. Triangle AED = triangle ETD [4]

c. Triangle AOE = triangle TOD [3]

33. In each of the following statements either too much or too little information has been given to justify the conclusion. Rewrite *each* statement so that the information given is just enough to justify the conclusion.

a. If the altitude and one side of an equilateral triangle are given, the triangle can be constructed. [2]

b. If the mid-points of the sides of a parallelogram are joined in order, the quadrilateral formed is a rhombus. [2]

c. If the angles of a triangle are in the ratio 1:2:3 and one of the angles is 30° more than another, the angles can be determined. [2]

d. If a teeterboard (seesaw) is 16 feet long and is supported at its center by a block 3 feet high, the height to which each end of the board can rise can be determined. [2]

e. If two circular disks each 1 inch in diameter and $\frac{1}{16}$ inch thick are melted and made into a single circular piece of metal of the same thickness, the ratio of the diameter of the new piece to the diameter of one of the given disks is $\sqrt{2}$:1 [2]

ANSWERS

EXERCISE 4

11. 72°; 72°; 108°.

EXERCISE 6

7. 90°.

EXERCISE 7

9. $25\frac{5}{7}°$; $64\frac{2}{7}°$.

EXERCISE 8

10. $25\frac{5}{7}°$; $154\frac{2}{7}°$. **11.** 60°.

EXERCISE 9

1. 40°; 60°; 80°. **2.** 80° 45′; 80° 45′; 18° 30′. **4.** 110°; $127\frac{1}{11}°$; $122\frac{1}{2}$. **10.** 8; 5. **11.** 6. **12.** No; yes; 72 sides.

EXERCISE 15

1. 5 in. and 11 in. **9.** 4 in. and 26 in.

EXERCISE 16

1. 5 in. **2.** 16 in. **3.** 14 in. **4.** 30°. **7.** $90\sqrt{3}$ sq. in. **10.** 3 in. **11.** 120 sq. in. **14.** 9 in. **15.** 6 in.

EXERCISE 21

12. $5\frac{1}{2}$ in.

EXERCISE 27

1. $77\frac{1}{2}°$; $102\frac{1}{2}°$; $91\frac{1}{2}°$; $88\frac{1}{2}°$; 84°. **2.** 56°. **5.** 150°. **6.** 105°; 255°. **7.** $81\frac{1}{2}°$. **8.** 105°; 75°. **9.** 65° *or* 115°. **10.** 84°. **14.** 144°. **15.** 110°.

EXERCISE 28

12. 54°. **13.** 58°; 62°; 60°. **15.** 60°; 84°; 20°; 96°.

EXERCISE 29

1. $3^{2}/_{11}$ ft.; $3^{9}/_{11}$ ft. **2.** $7^{5}/_{7}$; $10^{2}/_{7}$. **3.** 12. **4.** 10 in; 4 in. **5.** (*a*) *PA* and *PB*; (*b*) *PA* and *PB*. **6.** 24 yds.; 30 yds. **10.** $31^{1}/_{2}$ *or* 14. **11.** 2.4 in.; 5.6 in. **12.** 5 in.; 10 in.

EXERCISE 34

1. $7^{5}/_{7}$; $12^{6}/_{7}$. **2.** $4^{2}/_{7}$; $1^{5}/_{7}$; $2^{4}/_{7}$; $3^{3}/_{7}$. **3.** 6 in.; 2 in.; $7^{1}/_{2}$; $2^{1}/_{2}$ in **4.** 56 in. **5.** $9^{1}/_{3}$ in. **6.** 9.6; 15 *or* $3^{3}/_{4}$; $9^{3}/_{8}$ *or* $2^{2}/_{5}$; $3^{21}/_{25}$. **7.** Yes. **8.** $28^{4}/_{5}$. **9.** $16^{2}/_{3}$. **10.** $1^{1}/_{4}$ in.; $1^{1}/_{2}$ in. **11.** $2^{2}/_{3}$ in. **12.** 12 in.; 18 in.

EXERCISE 35

5. 9.6. **6.** 8.4 in.; 14.4 in. **7.** $6^{11}/_{19}$ in.; $18^{8}/_{19}$ in. **8.** $1^{24}/_{25}$; $23^{1}/_{25}$; 24. **9.** 9; 16; 15; 20. **11.** $9\sqrt{2}$.

EXERCISE 36

4. 9 in.; 16 in. **5.** $5^{2}/_{3}$ ft. **6.** 30 in. **7.** 460; 42.89 in. **8.** 10 in. **9.** 9 in. **10.** 4 ft.; 8 ft. **11.** 24. **12.** 216. **14.** 27 ft.

EXERCISE 37

3. 8. **4.** 24. **5.** $94^{2}/_{7}$. **6.** $12\sqrt{3}$ in. **8.** 50 ft. **9.** 1:6. **10.** 4:7.

EXERCISE 38

1. 20 in. **2.** 11.82 in. **3.** 3 ft.; 4 ft. **4.** 17 miles. **5.** 18 in.; 32 in.; 40 in. **6.** 30 in. **7.** $6\sqrt{3}$. **9.** $^{20}/_{3}\sqrt{3}$. **10.** 52. **11.** (*a*) obtuse; (*b*) acute; (*c*) obtuse; (*d*) right. **12.** $7\sqrt{2}$. **13.** 17. **14.** 20 ft. **15.** 5. **16.** 5. **17.** (*a*) 10.9; (*b*) $^{4}/_{7}$; (*c*) $73^{4}/_{7}$; (*d*) 0. **18.** $6\sqrt{11}$ *or* 19.90; $2\sqrt{51}$ *or* 14.28.

EXERCISE 39

1. 70 rds.; 56 rds. **2.** $2\sqrt{5}$; $7\sqrt{5}$. **4.** 720 sq. ft. **5.** 24; 384. **6.** 8 in. **7.** 6200 sq. ft. **8.** $120\sqrt{3}$. **9.** $128\sqrt{3}$ sq. in. **10.** 31,416 sq. ft. **12.** $45\sqrt{2}$ sq. in. **13.** 100. **14.** $4\sqrt{3}$. **15.** 108. **16.** $71^{1}/_{4}$. **17.** 300. **18.** 8 ft.; 21 ft. **20.** 147.41 sq. in.

EXERCISE 41

1. $96\sqrt{3}$ sq. in.

EXERCISE 43

1. 15 ft. **2.** 61.44 sq. in.; 19.2 in.; 15.36 in. **4.** 36. **5.** 28 ft. **6.** $3\sqrt{2}$; $4\sqrt{2}$. **7.** 3; 4. **9.** $\sqrt{2}$: 1. **10.** 20 sq. ft. **11.** 15 ft. **12.** 25: 64. **13.** 19.2; 97.2. **14.** 25: 64. **15.** $1\frac{9}{16}$ sq. ft.

EXERCISE 44

7. 27.73 ft. **8.** 310. **9.** 374. **10.** 46.69. **11.** 101.51. **13.** $\sqrt{3}$: 1 *or* $\sqrt{3}$:2. **16.** 17 in. **18.** 21.63 in.

EXERCISE 45

1. 120°; 90°; 72°; 60°; 45°; 36°. **4.** $6\sqrt{3}$.

EXERCISE 47

1. 0.7π; 1.4π. **2.** 33.85. **3.** $2\sqrt{14}$. **4.** 5.08. **5.** $43\frac{1}{5}°$. **6.** $1\frac{1}{2}$ ft. **7.** 30 ft.; 21.21 ft. **8.** $\frac{25\pi}{3}$ sq. in.; $\frac{100\pi}{3}$ sq. in. **9.** $1+\sqrt{2}$ *or* 2.41 ft. **10.** 144π sq. ft. **11.** 1425π sq. ft. **12.** 34π. **13.** 92.47. **14.** 1142.48. **15.** 0.65 sq. in. **16.** 5131.95 sq. in.

EXERCISE 48

1. (*a*) $16\sqrt{3}$; (*b*) $4\sqrt{3}$; (*c*) $\frac{8}{3}\sqrt{3}$; (*d*) $\frac{4}{3}\sqrt{3}$. **2.** $150\sqrt{3}$. **3.** 3: 4. **4.** 30; $25\sqrt{3}$. **5.** $27\sqrt{3}$ sq. in. **6.** 12. **7.** 6.08 ft. **8.** (*a*) $\frac{10}{3}\sqrt{3}$; (*b*) $\frac{5}{3}\sqrt{3}$. **10.** 3.26 sq. in. **11.** 2.46 sq. in. **12.** $6\frac{1}{3}$; $12\frac{2}{3}$. **13.** 26.1 sq. in.

EXERCISE 49

1. (*a*) $3\frac{1}{3}$; (*b*) $-3\frac{2}{3}$; *i.e.*, the triangle is obtuse. **2.** 210 sq. in.; 15 in. **3.** 12.97; 12.17; 11.24. **4.** $\frac{18}{7}\sqrt{15}$; $3\sqrt{6}$; $\frac{9}{8}\sqrt{6}$. **5.** $2\sqrt{2}$; $\frac{33}{8}\sqrt{2}$. **6.** 12 in. **7.** 16.8 sq. in. **8.** $4\sqrt{5}+4\sqrt{6}$. **9.** $3\sqrt{10}$; $\frac{3}{2}\sqrt{15}$ *or* $\frac{9}{4}\sqrt{15}$. **10.** 24. **11.** $\frac{1225\pi}{24}$; $\frac{35\pi\sqrt{6}}{6}$. **12.** $16\pi - 12\sqrt{3}$ *or* 29.48 sq. in.

EXERCISE 50

1. 229 ft. **2.** 8.94 in. **3.** 127.3 ft. **4.** 1.4 ft. **5.** $10\sqrt{3}$ in. **6.** 4; 16. **7.** Twice as much for first. **8.** (*a*) 40 ft.; (*b*) 96 sq. ft. **9.** $17\frac{1}{2}$ ft. **10.** 30 ft.

EXERCISE 51

3. 14 ft. **4.** $2\sqrt{21}$; $\sqrt{201}$; $\sqrt{129}$. **5.** 17. **6.** 20; 30. **7.** 24. **8.** 8. **9.** 15. **10.** 4 in. **11.** $6\frac{1}{4}$. **12.** 25 in. **13.** $2\pi + 6$. **14.** 150 sq. in. **15.** $\frac{21}{4}\sqrt{15}$ in. **16.** 26 in.; $9\frac{3}{13}$ in.

IMPORTANT FORMULAS

NOTATION USED

a, b, c = sides of triangle ABC.
p = perimeter.
r = radius of circle.
c = circumference of circle.
d = diameter.
h = altitude.
h_c = altitude on side c.
a = apothem of a regular polygon.
m_c = median on side c.
t_c = bisector of angle opposite c.
$s = \frac{1}{2}(a+b+c)$.
d = diameter of a circle, diagonal of a square.
K = area.
b = base of triangle, parallelogram, etc.
$\pi = \frac{22}{7}$ (approximately).

Length of Lines

Right triangle ($C = 90°$), $a^2+b^2=c^2$.

Equilateral triangle, $h = \frac{1}{2}b\sqrt{3}$.

In any triangle,

$$h_c = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}.$$

$$m_c = \frac{1}{2}\sqrt{2(a^2+b^2)-c^2}.$$

$$t_c = \frac{2}{a+b}\sqrt{abs(s-c)}.$$

Diagonal of square, $d = b\sqrt{2}$.

Side of square, $b = \sqrt{K}$.

In similar polygons, $p : p' = a : a'$, $a : a' = b : b'$.

Circle, $c = 2\pi r = \pi d$.

An arc of a circle, $\frac{\text{central angle}}{180°} \times \pi r$.

Areas of Plane Figures

Rectangle,	bh.	Trapezoid,	$\frac{1}{2}h(b_1+b_2)$.
Square,	b^2.	Regular polygon,	$\frac{1}{2}ap$.
Parallelogram,	bh.	Circle,	$\frac{1}{2}rc = \pi r^2$.
Equilateral triangle,	$\frac{1}{4}b^2\sqrt{3}$.	**Sector** of a circle,	$\frac{1}{2}r \times arc$.
Triangle,	$\frac{1}{2}bh$.		
or	$\sqrt{s(s-a)(s-b)(s-c)}$.	or	$\frac{\text{central angle}}{360°} \times \pi r^2$.

In a **segment** of a circle, K = sector $\pm \Delta$ formed by the chord and radii of the segment.

In any two similar plane figures,

$$K : K' = a^2 : a'^2.$$

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